Binary trees and (maximal) order types

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Abstract. Concerning the set of rooted binary trees, one shows that Higman's Lemma and Dershowitz's recursive path ordering can be used for the decision of its maximal order type according to the homeomorphic embedding relation as well as of the order type according to its canonical linearization, well-known in proof theory as the Feferman-Schütte notation system without terms for addition. This will be done by showing that the ordinal ω_{n+1} can be found as the (maximal) order type of a set in a cumulative hierarchy of sets of rooted binary trees.

1 Introduction

Well-partial-ordering: A *quasi-ordering* is a pair $(X, \preceq)$, where X is a set and $\preceq$ is a transitive, reflexive binary relation on X. If $Y \subseteq X$ we write $(Y, \preceq)$ instead of $(Y, \preceq \restriction Y \times Y)$. A quasi-ordering $(X, \preceq)$ is called a *partial ordering* if $\preceq$ is antisymmetric, too.

For any partial ordering $(X, \preceq)$ and any $x, y \in X$ we write $x \prec y$ for $x \preceq y$ and $y \not\preceq x$. A *linear ordering* is a partial ordering $(X, \preceq)$ in which any two elements are $\preceq$-comparable.

A *well-quasi-ordering* (*wqo*) is a quasi-ordering $(X, \preceq)$ such that there is no infinite sequence $\langle x_i \rangle_{i \in \omega}$ of elements of X satisfying: $x_i \not\preceq x_j$ for all $i < j$. A *well-partial-ordering* (*wpo*) is a partial ordering which is well-quasi-ordered. $(X, \prec)$ is called *well-ordering* if $(X, \preceq)$ is a linear *wpo*. The following condition is necessary and sufficient for a partial ordering $(X, \preceq)$ to be a *wpo*:

Every extension of $\preceq$ to a linear ordering on X is a well-ordering.

In the following, we assume a basic knowledge about ordinals up to ε_0 and their arithmetic. Here are some notations:

$$\omega_0(\alpha) := \alpha \qquad \omega_{n+1}(\alpha) := \omega^{\omega_n(\alpha)} \qquad \omega_n := \omega_n(1)$$

The *order type* of a well-ordering $(X, \prec)$, $otyp(\prec)$, is the least ordinal for which there is an order-preserving function $f : X \to \alpha$:

$$otyp(\prec) := \min\{\alpha\colon \text{there is an order-preserving function } f : X \to \alpha\}$$

Given a *wpo* $(X, \preceq)$ consider an extension $(X, \prec^+)$ which is a well-ordering. How big is the order type of the well-ordering? Is there any *non-trivial* upper bound for it? Here 'non-trivial' means that the bound is lower than the obvious upper bound obtained by considering the cardinality of X. To this question, de Jongh and Parikh [1] gives a clear answer.

Definition 1. *Given a wpo* $(X, \preceq)$ *its* maximal order type *is defined as follows:*

$$o(X, \preceq) := \sup\{otyp(\prec^+) : \prec^+ \text{ is a well-ordering on } X \text{ extending } \preceq\}.$$

We simply write $o(X)$ for $o(X, \preceq)$ if it causes no confusion.

Theorem 2 (de Jongh and Parikh [1]). *If* $(X, \preceq)$ *is a wpo, then there is a well-ordering* $\prec^+$ *on* X *extending* $\preceq$ *such that* $o(X) = otyp(\prec^+)$.

We refer to Schmidt [2] for more extensive study concerning maximal order type.

Higman embedding: Given a set A, let A^* be the set of finite sequences of elements from A. Let $(A, \preceq)$ be a partial ordering. The *Higman embedding* $\preceq_{\mathrm{H}}$ is the partial ordering on A^* defined as follows:

$$a_1, \ldots, a_m \preceq_{\mathrm{H}} b_1, \ldots, b_n$$

if there is a strictly increasing function $g : [1, m] \to [1, n]$ such that $a_i \preceq b_{g(i)}$ for all $i \in [1, m]$.

Theorem 3.

1. (Higman's Lemma) *If* $(A, \preceq)$ *is a wpo(resp. wqo), then so is* $(A^*, \preceq_{\mathrm{H}})$.
2. (de Jongh and Parikh) *If* $(A, \preceq)$ *is a wpo with* $o(A, \preceq) = \alpha > 0$*, then we have*:

$$o(A^*, \preceq_{\mathrm{H}}) = \begin{cases} \omega^{\omega^{\alpha-1}} & \text{if } \alpha \in \omega \setminus \{0\}. \\ \omega^{\omega^{\alpha}} & \text{if } \alpha = \beta + m, \text{ where } \beta \geq \omega,\ \beta \neq \omega^{\beta}, \text{ and } m \in \omega. \\ \omega^{\omega^{\alpha+1}} & \text{otherwise}. \end{cases}$$

Proof. See e.g. [3,1,2,4]. □

Binary trees: A *rooted binary tree* T is a set of nodes such that, if it is not empty, there is one distinguished node called the root of T and the remaining nodes are partitioned into two rooted binary trees. Here is a formal definition:

Assume a constant o and a binary function symbol φ are given. The set of rooted binary trees $\mathcal{B}$ is the least set of terms defined as follows:

- $o \in \mathcal{B}$;
- if α, $\beta \in \mathcal{B}$, then $\varphi(\alpha, \beta) \in \mathcal{B}$.

We will write $\varphi\alpha\beta$ instead of $\varphi(\alpha, \beta)$ if it causes no confusion. The *homeomorphic embeddability* relation $\trianglelefteq$ on $\mathcal{B}$ is the least subset of $\mathcal{B} \times \mathcal{B}$ defined as follows:

- $o \trianglelefteq \beta$ for all $\beta \in \mathcal{B}$;

– if $\alpha = \varphi\alpha_1\alpha_2$, $\beta = \varphi\beta_1\beta_2$, then $\alpha \trianglelefteq \beta$ if one of the following cases holds:
 (*i*) $\alpha \trianglelefteq \beta_1$ or $\alpha \trianglelefteq \beta_2$;
 (*ii*) $\alpha_1 \trianglelefteq \beta_1$ and $\alpha_2 \trianglelefteq \beta_2$.

Higman [3] showed that $(\mathcal{B}, \trianglelefteq)$ is a *wpo*, and in an unpublished paper, de Jongh showed that $o(\mathcal{B}, \trianglelefteq) = \varepsilon_0$. Furthermore, one easily finds a well-ordering $<$ extending $\trianglelefteq$ such that $otyp(<) = \varepsilon_0$: $\alpha < \beta$ is true if

– $\alpha = o$ and $\beta \neq o$; or
– $\alpha = \varphi\alpha_1\alpha_2$, $\beta = \varphi\beta_1\beta_2$ and one of the following cases holds:
 (*i*) $\alpha_1 < \beta_1$ and $\alpha_2 < \beta$; or
 (*ii*) $\alpha_1 = \beta_1$ and $\alpha_2 < \beta_2$; or
 (*iii*) $\alpha_1 > \beta_1$ and $\alpha \leq \beta_2$.

One can easily see that $\leq$ extends $\trianglelefteq$, and it is a folklore in proof theory that $<$ is a well-ordering on $\mathcal{B}$ with $otyp(<) = \varepsilon_0$. In fact, the system $(\mathcal{B}, <)$ is the system which is obtained from the Feferman-Schütte notation system for Γ_0 by omitting the addition terms. See e.g. [5,6,7,8] for more details.

In this paper, we will give a new proof that $o(\mathcal{B}, \trianglelefteq) = otyp(\mathcal{B}, <) = \varepsilon_0$. Furthermore, this will be done by characterizing the subsets of $\mathcal{B}$ which have ω_n as their maximal order types according to the homeomorphic embedding relation.

2 Cumulative hierarchies $(\mathcal{B}^d)_d$ and $(\mathcal{B}^{d,k})_k$

In Weiermann [9], a cumulative hierarchy of $\mathcal{B}^d$ sucht that $\bigcup_d \mathcal{B}_d = \mathcal{B}$ is presented. Here we give cumulative hierchies $(\mathcal{B}^{d,k})_k$ such that $\bigcup_k \mathcal{B}^{d,k} = \mathcal{B}^d$ for any $d > 0$.[1]

Given a natural number d we define $\mathcal{B}^d$ recursively as follows:

– $o \in \mathcal{B}^d$;
– if $d > 0$, $\alpha \in \mathcal{B}^{d-1}$, and $\beta \in \mathcal{B}^d$, then $\varphi\alpha\beta \in \mathcal{B}^d$.

And define $\rho^d(\alpha)$ for $\alpha \in \mathcal{B}$ as follows:

$$\rho^0(\alpha) = \alpha \quad \text{and} \quad \rho^{d+1}(\alpha) = \varphi\rho^d(\alpha)0$$

Lemma 4. *Let d be a natural number.*

1. $\mathcal{B} = \bigcup\{\mathcal{B}^d \colon d \in \omega\}$.
2. *If* $\alpha \in \mathcal{B}^d$*, then* $\alpha < \rho^{d+1}(o)$ *and* $\rho^k(\alpha) \in \mathcal{B}^{d+k}$.
3. $\rho^{d+1}(o) \in \mathcal{B}^{d+1} \setminus \mathcal{B}^d$.
4. *If* $\alpha < \beta$*, then* $\rho^d(\alpha) < \rho^d(\beta)$.

[1] These cumulative hierarchies are essential for the proofs of phase transition of some combinatorial properties with respect to PA or $\mathrm{I}\Sigma_n$ respectively since they allow one to a structural approach to the sets from below. See Weiermann [9] and Lee [10] for more about phase transition concerning binary trees.

5. *If* $\alpha \trianglelefteq \beta$, *then* $\rho^d(\alpha) \trianglelefteq \rho^d(\beta)$.
6. *If* $\alpha \in \mathcal{B}^{d+1} \setminus \mathcal{B}^d$ *and* $\beta \in \mathcal{B}^d$, *then* $\alpha \ntrianglelefteq \beta$ *and* $\beta < \alpha$.

Proof. The first five claims are obvious. We show the last assertion by induction on α and β. If $\beta = 0$ there is nothing to show. Let $\alpha = \varphi\alpha_1\alpha_2$ and $\beta = \varphi\beta_1\beta_2$. If $\alpha_1 \in \mathcal{B}^{d-1}$, then $\alpha_2 \in \mathcal{B}^{d+1} \setminus \mathcal{B}^d$. Hence $\beta < \alpha_2 < \alpha$ by I.H. Now assume $\alpha_1 \in \mathcal{B}^d \setminus \mathcal{B}^{d-1}$. Then $\beta_1 < \alpha_1$ and $\beta_2 < \alpha$ by I.H., so $\beta < \alpha$ and $\alpha \ntrianglelefteq \beta$. □

Note that ω and B^1 can be identified by the isomorphism f defined as follows: $f(0) := o$ and $f(n+1) := \varphi(o, f(n))$. Hence we may talk about occurrences of natural numbers in $\alpha \in \mathcal{B}^d$, $d \geq 1$.

For $k \geq 1$ define

- $\mathcal{B}^{1,k} := \{0, 1, \ldots, k-1\}$.
- $\mathcal{B}^{d+1,k} := \{\alpha\colon \alpha = 0 \text{ or } \alpha = \varphi\beta\gamma, \text{ where } \beta \in \mathcal{B}^{d,k} \text{ and } \gamma \in \mathcal{B}^{d+1,k}\}$.

Lemma 5. *Let* d, k *be natural numbers.*

1. $\mathcal{B}^d = \bigcup_{k>0} \mathcal{B}^{d,k}$.
2. *If* $\alpha \in \mathcal{B}^{d+1,k}$, *then* $\alpha < \rho^d(k)$.
3. *If* $\alpha \in \mathcal{B}^{d,k+1} \setminus \mathcal{B}^{d,k}$ *and* $\beta \in \mathcal{B}^{d,k}$, *then* $\beta < \alpha$ *and* $\alpha \ntrianglelefteq \beta$.

Proof. Every claim can be shown by an simple induction on k. □

Given a positive natural number n define $\mathcal{B}_n$ by

$$\mathcal{B}_n := \begin{cases} \mathcal{B}^{d+1} & \text{if } n = 2d \\ \mathcal{B}^{d+1,2} & \text{if } n = 2d-1\,. \end{cases}$$

We claim

$$o(\mathcal{B}_n, \trianglelefteq \restriction \mathcal{B}_n) = otyp(< \restriction \mathcal{B}_n) = \omega_{n+1}\,.$$

3 Maximal order types

In general it is not a simple task to decide the maximal order type of a *wpo*. Some interesting methods are introduced in [2,11,4]. However, there is a problem that in most cases they can be carried out in a long-winded way only. Fortunately, there is a much more simple way for our case. We are going to take a well-known *wpo* and compare it with $(\mathcal{B}_n, \trianglelefteq)$.

Note first that the two sets $\mathcal{B}^{d+1}$ and $(\mathcal{B}^d)^*$ are similarly constructed. In fact, every $\alpha \in \mathcal{B}^{d+1}$ is of the form $\alpha = \varphi\alpha_1\varphi\alpha_2\cdots\varphi\alpha_m o$, where $\alpha_i \in \mathcal{B}^d$. If $\beta = \varphi\beta_1\varphi\beta_2\cdots\varphi\beta_n o \in \mathcal{B}^{d+1}$ and $\alpha_1\cdots\alpha_m \trianglelefteq_{\mathrm{H}} \beta_1\cdots\beta_n$ then $\alpha \trianglelefteq \beta$. And, though this relationship is not isomorphic, we can in fact show that $o(\mathcal{B}^{d+1}, \trianglelefteq) = o((\mathcal{B}^d)^*, \trianglelefteq_{\mathrm{H}})$.

We need the following obvious fact.

Lemma 6. *Let* $(A, \preceq_1)$ *and* $(B, \preceq_2)$ *be wpo's and* $f\colon A \to B$ *an injective function such that*

$$a \preceq_1 b \iff f(a) \preceq_2 f(b)$$

for all $a, b \in A$. *Then it holds that* $o(A) \leq o(B)$.

Theorem 7. *For any* $d > 0$, $o(\mathcal{B}^{d+1}, \trianglelefteq) = o(\mathcal{B}^{d+1} \setminus \mathcal{B}^d, \trianglelefteq) = o((\mathcal{B}^d)^*, \trianglelefteq_{\mathrm{H}})$.

Proof. Define $f\colon \mathcal{B}^{d+1} \to (\mathcal{B}^d)^*$ and $g\colon (\mathcal{B}^d)^* \to \mathcal{B}^{d+1} \setminus \mathcal{B}^d$ defined as follows:

$$f(\alpha) := \begin{cases} \epsilon & \text{if } \alpha = o \\ \alpha & \text{if } \alpha = \varphi\alpha_1\alpha_2 \in \mathcal{B}^d \\ \alpha_1, f(\alpha_2) & \text{if } \alpha = \varphi\alpha_1\alpha_2 \notin \mathcal{B}^d \end{cases}$$

and

$$g(\alpha_1, \ldots, \alpha_m) := \varphi\alpha_1\varphi\alpha_2 \cdots \varphi\alpha_m\rho^{d+1}(o)$$

where ϵ denotes the empty sequence. It is then very easy to show that f and g satisfy the conditions in Lemma 6. So we have the desired equalities. □

Corollary 8. *For any* $d > 0$, $(\mathcal{B}^d, \trianglelefteq)$ *is a wpo and* $o(\mathcal{B}^d, \trianglelefteq) = \omega_{2d-1}$.

Proof. By induction on $d > 0$. If $d = 1$, then $\mathcal{B}^1 = \{o, \varphi oo, \varphi o(\varphi oo), \ldots\}$ is linearly ordered by $\trianglelefteq$ and so $o(\mathcal{B}^1, \trianglelefteq) = \omega$. If $d > 1$, use I.H., Theorem 7, and Theorem 3. □

Corollary 9. $(\mathcal{B}, \trianglelefteq)$ *is a well-ordering and* $o(\mathcal{B}) = \varepsilon_0$.

Lemma 10. *Let* d, k *be positive natural numbers. Then*

$$o(\mathcal{B}^{d,k}, \trianglelefteq \restriction \mathcal{B}^{d,k}) = \begin{cases} k & \text{if } d = 1 \\ \omega_{2(d-1)}(k-1) & \text{otherwise}. \end{cases}$$

Proof. Similar to Corollary 8 □.

Theorem 11. $o(\mathcal{B}_n, \trianglelefteq \restriction \mathcal{B}_n) = \omega_{n+1}$ *for any positive natural number* n.

4 Order types

We are now going to compute the order types of $(\mathcal{B}_n, < \restriction \mathcal{B}_n)$. It is not so obvious as it might seem. $\mathcal{B}$ will be considered as ordinal notation systems based on the recursive path ordering on strings.

Definition 12 (Recursive path ordering). *Let* $(\mathcal{A}, \prec)$ *be a well-ordering. The* recursive path ordering $\prec_{rpo}$ *on* $\mathcal{A}^*$ *is defined as follows: Let* ϵ *be the empty list.*

- *If* $\epsilon \prec_{rpo} u$ *for* $u \neq \epsilon$.
- *If* $u = au_1$ *and* $v = bv_1$, *then* $u \prec_{rpo} v$ *if one of the following holds:*

(i) $a \prec b$ *and* $u_1 \prec_{rpo} v$*;*
(ii) $a = b$ *and* $u_1 \prec_{rpo} v_1$*;*
(iii) $b \prec a$ *and* $u \preceq_{rpo} v_1$*.*

Dershowitz [12] shows that the recursive path ordering preserves the well-orderedness.

Theorem 13 (Dershowitz). *If* $(\mathcal{A}, \prec)$ *is a well-ordering, so is* $(\mathcal{A}^*, \prec_{rpo})$*.*

Let $\xi < \varepsilon_0$ be the order type of $\prec$ on $\mathcal{A}$ and $\eta \mapsto a_\eta$, $\eta < \xi$, the enumeration function of $\mathcal{A}$. Using the idea elaborated by Touzet [13] we are going to characterize the order type of $\prec_{rpo}$ on $\mathcal{A}^*$.

Lemma 14. *For each limit ordinal* $\alpha < \omega^{\omega^\xi}$ *there are unique* γ*,* β*, and* $\eta < \xi$ *such that*

(i) $\alpha = \gamma + \omega^{\omega^\eta} \cdot \beta$*,*
(ii) $0 < \beta < \omega^{\omega^{\eta+1}}$*, and*
(iii) *there are no* $\mu \in \omega^{\omega^{\eta+1}} \setminus \{0\}$ *and* $\delta \in \omega^{\omega^\xi}$ *such that* $\gamma = \delta + \mu$*.*

Proof. Let $\alpha =_{NF} \omega^{\alpha_0} + \cdots + \omega^{\alpha_n}$. Let $\eta < \xi$ and j be such that

$$\omega^\eta \le \alpha_n < \omega^{\eta+1} \quad \text{and} \quad j := \min\{k \colon \omega^\eta \le \alpha_k < \omega^{\eta+1}\}\,.$$

There are δ_k, $j \le k \le n$, such that $\alpha_j = \omega^\eta + \delta_j, \ldots, \alpha_n = \omega^\eta + \delta_n$. Hence $\alpha = \gamma + \omega^{\omega^\eta} \cdot \beta$, where $\gamma =_{NF} \omega^{\alpha_0} + \cdots + \omega^{\alpha_{j-1}}$ and $\beta =_{NF} \omega^{\delta_j} + \cdots + \omega^{\delta_n}$, and η, β, γ satisfy the conditions (ii) and (iii).

We now prove the uniqueness of the decomposition. Let η', β', γ' also satisfy (i) $\sim$ (iii). If $\beta =_{NF} \omega^{\beta_0} + \cdots + \omega^{\beta_m}$ and $\beta' =_{NF} \omega^{\beta'_0} + \cdots + \omega^{\beta'_\ell}$ and if γ is in Cantor normal form too, then conditions (ii) and (iii) guarantee that

$$\alpha =_{NF} \gamma + \omega^{\omega^\eta + \beta_1} + \cdots + \omega^{\omega^\eta + \beta_m} =_{NF} \gamma' + \omega^{\omega^{\eta'} + \beta'_1} + \cdots + \omega^{\omega^{\eta'} + \beta'_\ell}$$

and hence $\eta = \eta'$. Suppose for instance $\gamma < \gamma'$. Then $\gamma' = \gamma + \omega^{\omega^\eta + \beta_1} + \cdots + \omega^{\omega^\eta + \beta_p}$ for some $p \le m$. This contradicts (iii). So $\gamma = \gamma'$ and hence $m = \ell$, $\beta_k = \beta'_k$, $1 \le k \le m$. □

In the sequel, $\gamma + \omega^{\omega^\eta} \cdot \beta$ means always in the sense of Lemma 14. For ordinals $\beta > 0$, $-1 + \beta$ denotes $\beta - 1$ if $\beta < \omega$ and β otherwise.

Definition 15. *Let* $(\mathcal{A}, \prec)$ *be a well-ordering and* $otyp(\prec) = \xi \in \varepsilon_0 \setminus \{0\}$*. The function* $\mathcal{O} \colon \omega^{\omega^{-1+\xi}} \to \mathcal{A}^*$ *is defined by:*

$$\mathcal{O}(\alpha) := \begin{cases} \epsilon & \text{if } \alpha = 0 \\ a_0 \mathcal{O}(\beta) & \text{if } \alpha = \beta + 1 \\ a_{1+\eta} \mathcal{O}(-1+\beta)\mathcal{O}(\gamma) & \text{if } \alpha = \gamma + \omega^{\omega^\eta} \cdot \beta\,. \end{cases}$$

Now we are going to show that the definition of $((B^d)^*, <_{rpo})$ is just another way to see $(B^{d+1}, <)$.

Theorem 16. *Let* $(\mathcal{A}, \prec)$ *be a well-ordering. If* $otyp(\prec) = \xi \in \varepsilon_0 \setminus \{0\}$ *on* $\mathcal{A}$*, then we have on* $\mathcal{A}^*$

$$otyp(\prec_{rpo}) = \omega^{\omega^{\omega^{-1+\xi}}} = \begin{cases} \omega^{\omega^{\omega^{\xi-1}}} & \textit{if } \xi \in \omega \setminus \{0\} \\ \omega^{\omega^{\omega^{\xi}}} & \textit{otherwise}. \end{cases}$$

Proof. We show that the function $\mathcal{O}\colon (\omega^{\omega^{\omega^{-1+\xi}}}, <) \to (\mathcal{A}^*, \prec_{rpo})$ is an isomorphism.

1. $\mathcal{O}$ is order-preserving, i.e. $\mathcal{O}(\alpha) \prec_{rpo} \mathcal{O}(\beta)$ if $\alpha < \beta$. Note that the ordering $<$ on ordinals is the transitive closure of the schemes $\forall n \in \omega(\alpha_n < \alpha)$, where $(\alpha_n)_n$ builds a fundamental sequence for α. (The definition of the fundamental sequence will be directly given below in the proof.) So it suffices to show that $\forall n \in \omega(\mathcal{O}(\alpha_n) \prec_{rpo} \mathcal{O}(\alpha))$ for any $\alpha < \xi$.
   (a) $\alpha = \beta + 1$: Then $\alpha_n = \beta$ and $\mathcal{O}(\alpha_n) = \mathcal{O}(\beta) \prec_{rpo} a_0\mathcal{O}(\beta) = \mathcal{O}(\alpha)$.
   (b) $\alpha = \gamma + \omega^{\omega^{\eta}} \cdot (\beta + 1)$:
   – $\eta = 0$, i.e. $\alpha_n = \gamma + \omega^{\omega^0} \cdot \beta + n + 1$: Then

$$\mathcal{O}(\alpha_n) = \begin{cases} a_0^{n+1}\mathcal{O}(\gamma) & \text{if } \beta = 0 \\ a_0^{n+1}a_1\mathcal{O}(-1+\beta)\mathcal{O}(\gamma) & \text{otherwise} \end{cases}$$

$$\prec_{rpo}$$

$$\mathcal{O}(\alpha) = \begin{cases} a_1\mathcal{O}(\gamma) & \text{if } \beta = 0 \\ a_1\mathcal{O}(-1+\beta+1)\mathcal{O}(\gamma) & \text{otherwise}. \end{cases}$$

   – $\eta = \eta_0 + 1$, i.e. $\alpha_n = \gamma + \omega^{\omega^{\eta}} \cdot \beta + \omega^{\omega^{\eta_0}} \cdot \omega^{\omega^{\eta_0} \cdot n}$: Then

$$\mathcal{O}(\alpha_n) = \begin{cases} a_{1+\eta_0}^{n+1}\mathcal{O}(\gamma) & \text{if } \beta = 0 \\ a_{1+\eta_0}^{n+1}a_{1+\eta}\mathcal{O}(-1+\beta)\mathcal{O}(\gamma) & \text{otherwise} \end{cases}$$

$$\prec_{rpo}$$

$$\mathcal{O}(\alpha) = \begin{cases} a_{1+\eta}\mathcal{O}(\gamma) & \text{if } \beta = 0 \\ a_{1+\eta}\mathcal{O}(-1+\beta+1)\mathcal{O}(\gamma) & \text{otherwise}. \end{cases}$$

   – η is a limit ordinal, i.e.$\alpha_n = \gamma + \omega^{\omega^{\eta}} \cdot \beta + \omega^{\omega^{\eta_n}}$: Then

$$\mathcal{O}(\alpha_n) = \begin{cases} a_{1+\eta_n}\mathcal{O}(\gamma) & \text{if } \beta = 0 \\ a_{1+\eta_n}a_{\eta}\mathcal{O}(-1+\beta)\mathcal{O}(\gamma) & \text{otherwise} \end{cases}$$

$$\prec_{rpo}$$

$$\mathcal{O}(\alpha) = \begin{cases} a_{\eta}\mathcal{O}(\gamma) & \text{if } \beta = 0 \\ a_{\eta}\mathcal{O}(-1+\beta+1)\mathcal{O}(\gamma) & \text{otherwise}. \end{cases}$$

   (c) $\alpha = \gamma + \omega^{\omega^{\eta}} \cdot \lambda$, where λ is a limit ordinal: Then $\alpha_n = \gamma + \omega^{\omega^{\eta}} \cdot \lambda_n$ and $\mathcal{O}(\alpha_n) = a_{1+\eta}\mathcal{O}(-1+\lambda_n)\mathcal{O}(\gamma) \prec_{rpo} a_{1+\eta}\mathcal{O}(-1+\lambda)\mathcal{O}(\gamma) = \mathcal{O}(\alpha)$.

We have shown that $\mathcal{O}$ is order-preserving, so it is injective.

2. Let $u \in \mathcal{A}^*$. By induction on the length of u we show that there is an $\alpha < \omega^{\omega^\xi}$ such that $\mathcal{O}(\alpha) = u$.
   (a) $u = \epsilon$: $\mathcal{O}(0) = \epsilon$.
   (b) $u = a_0 v$: Then $\mathcal{O}(\beta + 1) = a_0 v$, where $\mathcal{O}(\beta) = v$.
   (c) $u = a_\eta v$, $\eta > 0$: Then let $\eta' = \eta$ if $\eta \geq \omega$ and $\eta' = \eta + 1$ otherwise.
   - $v \in \{a_0, \ldots, a_\eta\}^*$: Let $\mathcal{O}(-1 + \beta) = v$. Then $-1 + \beta < \omega^{\omega^{\eta'}}$ and $\mathcal{O}(\omega^{\omega^{-1+\eta}} \cdot \beta) = a_\eta \mathcal{O}(-1 + \beta) = a_\eta v = u$.
     Note that this case implies, in particular, that $\mathcal{O}\colon \omega^{\omega^{\xi-1}} \to \mathcal{A}^*$ is an isomorphism if $\xi \in \omega \setminus \{0\}$. Indeed, if $\mathcal{A} = \{a_0, \ldots, a_\eta\}$ and $\xi = \eta + 1$ then we have just shown that $\alpha < \omega^{\omega^\eta}$ for α such that $\mathcal{O}(\alpha) = u$.
   - $v \notin \{a_0, \ldots, a_\eta\}^*$: Let $b \in \mathcal{A} \setminus \{a_0, \ldots, a_\eta\}$, $v_1 \in \{a_0, \ldots, a_\eta\}^*$, and $v_2 \in \mathcal{A}^*$ such that $v = v_1 b v_2$. Let $\mathcal{O}(-1 + \beta) = v_1$ and $\mathcal{O}(\gamma) = b v_2$. Then $\mathcal{O}(\gamma + \omega^{\omega^{-1+\eta}} \cdot \beta) = a_\eta \mathcal{O}(-1 + \beta)\mathcal{O}(\gamma) = a_\eta v_1 b v_2 = u$. □

Corollary 17. *For any $d > 0$, $(\mathcal{B}^d, <)$ is a well-ordering and otyp$(< \restriction \mathcal{B}^d) = \omega_{2d-1}$.*

Proof. Note just that $(\mathcal{B}^{d+1}, <)$ is isomorphic to $((\mathcal{B}^d)^*, <_{rpo})$. □

Corollary 18. *$(\mathcal{B}, <)$ is a well-ordering and otyp$(<) = \varepsilon_0$.*

Lemma 19. *Let d, k be positive natural numbers. Then*

$$otyp(< \restriction \mathcal{B}^{d,k}) = \begin{cases} k & \text{if } d = 1 \\ \omega_{2(d-1)}(k-1) & \text{otherwise}. \end{cases}$$

Proof. Similar to Corollary 17. □

Theorem 20. *otyp$(\mathcal{B}_n, < \restriction \mathcal{B}_n) = \omega_{n+1}$ for any positive natural number n.*

Finally, Theorem 11 and Theorem 20 imply the main claim.

Theorem 21. *$o(\mathcal{B}_n, \trianglelefteq \restriction \mathcal{B}_n) =$ otyp$(< \restriction \mathcal{B}_n) = \omega_{n+1}$ for any positive natural number n.*

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