Well-partial-orderings and hierarchies of binary trees

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Abstract

The set of binary trees together with the homeomorphic embedding build an well-partial-ordering. It is well known that its maximal order type is ε_0 as shown by de Jongh and that its canonical linearization is the system obtained from the celebrated Feferman-Schütte system for Γ_0 by omitting the addition operator. We show here that Higman's Lemma and Dershowitz's recursive path ordering are nice tools in deciding the maximal order type and the order type, respectively. Moreover, this will be done by showing that the ordinal $\omega_n(k)$ can be found as the (maximal) order type of a set in a cumulative hierarchy of binary trees.

Key words: Well-partial-ordering, binary trees, ordinal notation systems
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1 Introduction

The notion of well-partial-ordering appeared around 1950. After the celebrated work by Higman [1], this simple notion has found a large number of applications in the fields of algebras, combinatorics, mathematical logic, and computer science. One of the most elegant is Kruskal's theorem.

In the 80's, Friedman used it to give independence results for some segments of second order arithmetic [2,3]. For more about its meaning and importance we refer to Kolata [4] which provides an overview of the impact of Gödel's theorems on mathematics. On the other hand, Dershowitz showed how useful it could be in proving the termination of programs in computer science [5].

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Another important thing about well-partial-orderings is the maximal order type, i.e. the greatest order type of linearizations. This has an immediate consequence that well-partial-orderings are directly connected with ordinal notation systems. This is because of the equivalence of well-partial-orderedness and well-orderedness of ordinals. Since Gentzen's consistency proof of Peano arithmetic [6], it is well known that well-orderedness of large ordinals is independent of some logical systems. For example, that of the ordinal ε_0 is independent of the systems PA and ACA_0. This implies that the well-partial-orderedness of any well-partial-ordering of maximal order type ε_0 is independent too. An example is $(\mathcal{B}, \trianglelefteq)$, the set of binary trees with the homeomorphic embedding. Higman [1] showed that $(\mathcal{B}, \trianglelefteq)$ is a *wpo*, and in an unpublished paper, de Jongh showed that its maximal order type is ε_0, cf. Schmidt [7].

The main point of this paper is to give a new proof of the fact mentioned just before by considering the set of binary trees as a cumulative hierarchy of well-partial-orderings. We use Higman's lemma to decide their maximal order types. Furthermore, Derschowitz's recursive path ordering will be used for the decision of the order types of the sets from the hierarchy when they are canonically linearized. Here *canonically linearized* means that they are sub-orderings of a celebrated well-ordering in proof theory.

This paper is an extended version of the proceedings paper Lee [8] of CiE 2007. It contains more general introduction about well-partial-orderings, more detailed proofs and some new results. The new results are about the characterization of ordinals $\omega_n(k)$ using binary trees and included in the last section.

The rest of this section gives basic definitions and some important properties.

Well-partial-ordering A *quasi-ordering* is a pair $(X, \preceq)$, where X is a set and $\preceq$ is a transitive, reflexive binary relation on X. If $Y \subseteq X$ we write $(Y, \preceq)$ instead of $(Y, \preceq \restriction Y \times Y)$. A quasi-ordering $(X, \preceq)$ is called a *partial ordering* if $\preceq$ is antisymmetric, too.

For any partial ordering $(X, \preceq)$ and any x, $y \in X$ we write $x \prec y$ for $x \preceq y$ and $y \not\preceq x$. A *linear ordering* (total ordering) is a partial ordering $(X, \preceq)$ in which any two elements are $\preceq$-comparable.

A *well-quasi-ordering* (*wqo*) is a quasi-ordering $(X, \preceq)$ such that there is no infinite sequence $\langle x_i \rangle_{i \in \omega}$ of elements of X satisfying: $x_i \not\preceq x_j$ for all $i < j$. A *well-partial-ordering* (*wpo*) is a partial ordering which is well-quasi-ordered. $(X, \prec)$ is called *well-ordering* if $(X, \preceq)$ is a linear *wpo*. The following condition is necessary and sufficient for a partial ordering $(X, \preceq)$ to be a *wpo*:

Every extension of $\preceq$ to a linear ordering on X is a well-ordering.

In the following, we assume a basic knowledge about ordinals up to ε_0 and their arithmetic. Here are some notations:

$$\omega_0(\alpha) := \alpha \qquad \omega_{n+1}(\alpha) := \omega^{\omega_n(\alpha)} \qquad \omega_n := \omega_n(1)$$

The *order type* of a well-ordering $(X, \prec)$, $otyp(\prec)$, is the least ordinal for which there is an order-preserving function $f : X \to \alpha$:

$$otyp(\prec) := \min\{\alpha \,|\, \text{there is an order-preserving function } f : X \to \alpha\}$$

Given a *wpo* $(X, \preceq)$ consider an linear extension $(X, \prec^+)$. How big is the order type of the well-ordering? Is there any non-trivial upper bound for it? Here *non-trivial* means that the bound is lower than the obvious upper bound obtained by considering the cardinality of X. To this question, de Jongh and Parikh [9] gives a clear answer: Given a *wpo* $(X, \preceq)$ its *maximal order type* is defined as follows:

$$o(X, \preceq) := \sup\{otyp(\prec^+) \,|\, \prec^+ \text{ is a well-ordering on } X \text{ extending } \preceq\}\,.$$

We simply write $o(X)$ for $o(X, \preceq)$ if it causes no confusion.

Theorem 1 (de Jongh and Parikh [9]) *If $(X, \preceq)$ is a wpo, then there is a well-ordering $\prec^+$ on X extending $\preceq$ such that $o(X) = otyp(\prec^+)$.*

We refer to Schmidt [7] for more extensive study about maximal order type. We mention just a very useful fact:

Lemma 2 *If $(X, \preceq_X)$ and $(Y, \preceq_Y)$ are wpo, so are $(X + Y, \preceq_X + \preceq_Y)$ and $(X \times Y, \preceq_X \times \preceq_Y)$. In addition, we have*

$$o(X + Y) = o(X) \oplus o(Y) \quad \text{and} \quad o(X \times Y) = o(X) \otimes o(Y)\,.$$

where $\oplus$ (resp. $\otimes$) denotes natural sum (resp. natural product) and $+$ (resp. $\times$) denotes disjoint union (resp. Cartesian product).

PROOF. See Schmidt [7]. □

Kruskal's theorem and slowly-well-orderedness Friedman [2] gives an independence result which has attracted the attention of mathematicians: the so-called *Friedman-style miniaturization* of Kruskal's theorem about finite trees. A *finite tree* T is a finite partial ordering $(T, \preceq)$ such that, if T is not empty, there is a smallest element called the *root* of T and that, for each $b \in T$, the set $\{a \in T \,|\, a \preceq b\}$ is totally ordered. Let $a \wedge b$ denote the infimum of a and b for a, $b \in T$. A finite tree T_1 is called *homeomorphically embeddable* into a finite tree T_2, $T_1 \trianglelefteq T_2$, if there is an injection $f \colon T_1 \to T_2$ such that $f(a \wedge b) = f(a) \wedge f(b)$ for all a, $b \in T_1$. *Kruskal's theorem* says:

The set of finite trees with the homeomorphic embedding $\trianglelefteq$ build a wpo.

This is a true sentence, cf. Kruskal [10], but Friedman showed that it is ATR_0-independent. The formula complexity of Kruskal's theorem is Π^1_1, i.e., it does not belong to the language of first-order arithmetic. By the Friedman-style miniaturization, however, it can be transformed into a sentence of a relatively simple complexity. This is done with the help of a certain norm function:

For any natural number k, there is a constant n so large that, for any finite sequence $T_0, \ldots, T_n$ of finite trees such that the number of nodes of T_i is at most $k + f(i)$ for all $i \leq n$, there are indices $\ell < m \leq n$ such that T_ℓ is homeomorphically embeddable into T_m.

Note that this is a Π^0_2 sentence if $f \colon \mathbb{N} \to \mathbb{N}$ is primitive recursive, i.e., it isn't more mathematically complicated than its infinite prototype. Let $\mathrm{SWP}(\mathbb{T}, \trianglelefteq, f)$ denote this Π^0_2 sentence.

Theorem 3

(1) (Friedman [2], Smith [11]) $\mathrm{ATR}_0 \nvdash \mathrm{SWP}(\mathbb{T}, \trianglelefteq, id)$ *.*
(2) (Loebl and Matoušek [12]) *Let $f_r(i) = r \cdot \log_2(i)$. Then*

$$\mathrm{PA} \nvdash \mathrm{SWP}(\mathbb{T}, \trianglelefteq, f_4) \quad \text{and} \quad \mathrm{PA} \vdash \mathrm{SWP}(\mathbb{T}, \trianglelefteq, f_{\frac{1}{2}})\,.$$

Higman embedding Given a set A, let A^* be the set of finite sequences of elements from A. Let $(A, \preceq)$ be a partial ordering. The *Higman embedding* $\preceq_{\mathrm{H}}$ is the partial ordering on A^* defined as follows:

$$a_1, \ldots, a_m \preceq_{\mathrm{H}} b_1, \ldots, b_n$$

if there is a strictly increasing function $g : [1, m] \to [1, n]$ such that $a_i \preceq b_{g(i)}$ for all $i \in [1, m]$.

Theorem 4

(1) (Higman's Lemma) *If $(A, \preceq)$ is a wpo(resp. wqo), then so is $(A^*, \preceq_{\mathrm{H}})$.*
(2) (de Jongh and Parikh) *If $(A, \preceq)$ is a wpo with $o(A, \preceq) = \alpha > 0$, then we have*:

$$o(A^*, \preceq_{\mathrm{H}}) = \begin{cases} \omega^{\omega^{\alpha-1}} & \text{if } \alpha \in \omega \setminus \{0\}\,. \\ \omega^{\omega^{\alpha}} & \text{if } \alpha = \beta + m, \text{ where } \beta \geq \omega,\ \beta \neq \omega^\beta, \text{ and } m \in \omega\,. \\ \omega^{\omega^{\alpha+1}} & \text{otherwise}\,. \end{cases}$$

PROOF. See e.g. [1,9,7,13]. $\square$

We now give a short overview of this paper. In Section 2, a cumulative hierarchy of binary trees will be defined. The well-partial-orderedness of the sets from the hierarchy and their maximal order types will be decided in Section 3. We show in Section 4 how the recursive path ordering can be used for the decision of the order types of the canonical linearization of the well-partial-orderings. The last section explains briefly the starting point of this paper and give a hierarchy of well-partial-orderings with maximal order types $\omega_n(k)$.

2 Binary trees and well-partial-orderedness

A *binary tree* T is a set of nodes such that, if it is not empty, there is one distinguished node, the *root*, and the remaining nodes are partitioned into two binary trees. Here is a formal definition:

Assume a constant o and a binary function symbol φ are given. The set of binary trees $\mathcal{B}$ is the least set of terms defined as follows:

- $o \in \mathcal{B}$;
- if α, $\beta \in \mathcal{B}$, then $\varphi(\alpha, \beta) \in \mathcal{B}$.

We will write $\varphi\alpha\beta$ instead of $\varphi(\alpha, \beta)$. The *homeomorphic embeddability* relation $\trianglelefteq$ on $\mathcal{B}$ is the least subset of $\mathcal{B} \times \mathcal{B}$ defined as follows:

- $o \trianglelefteq \beta$ for all $\beta \in \mathcal{B}$;
- if $\alpha = \varphi\alpha_1\alpha_2$, $\beta = \varphi\beta_1\beta_2$, then $\alpha \trianglelefteq \beta$ if one of the following cases holds:

(1) $\alpha \trianglelefteq \beta_1$ or $\alpha \trianglelefteq \beta_2$;
(2) $\alpha_1 \trianglelefteq \beta_1$ and $\alpha_2 \trianglelefteq \beta_2$.

Higman [1] showed that $(\mathcal{B}, \trianglelefteq)$ is a *wpo*, and in an unpublished paper, de Jongh showed that $o(\mathcal{B}, \trianglelefteq) = \varepsilon_0$. Furthermore, one easily finds a well-ordering $<$ extending $\trianglelefteq$ such that $otyp(<) = \varepsilon_0$: $\alpha < \beta$ is true if

- $\alpha = o$ and $\beta \neq o$;
- $\alpha = \varphi\alpha_1\alpha_2$, $\beta = \varphi\beta_1\beta_2$ and one of the following cases holds:

(1) $\alpha_1 < \beta_1$ and $\alpha_2 < \beta$;
(2) $\alpha_1 = \beta_1$ and $\alpha_2 < \beta_2$;
(3) $\alpha_1 > \beta_1$ and $\alpha \leq \beta_2$.

$\leq$ obviously extends $\trianglelefteq$, and it is a folklore in proof theory that $<$ is a well-ordering on $\mathcal{B}$ such that $otyp(<) = \varepsilon_0$. In fact, the system $(\mathcal{B}, <)$ is the system obtained from the Feferman-Schütte notation system for Γ_0 by omitting the addition terms. See e.g. [14–17] for more details.

In Weiermann [18], a cumulative hierarchy of $\mathcal{B}^d$ such that $\bigcup_d \mathcal{B}_d = \mathcal{B}$ is presented. We also define cumulative hierarchies $(\mathcal{B}^{d,k})_k$ such that $\bigcup_k \mathcal{B}^{d,k} = \mathcal{B}^d$ for any $d > 0$.

Given a natural number d, we define $\mathcal{B}^d$ recursively:

- $o \in \mathcal{B}^d$;
- if $d > 0$, $\alpha \in \mathcal{B}^{d-1}$, and $\beta \in \mathcal{B}^d$, then $\varphi\alpha\beta \in \mathcal{B}^d$.

Define also $\rho^d(\alpha)$ for $\alpha \in \mathcal{B}$ as follows:

$$\rho^0(\alpha) = \alpha \quad \text{and} \quad \rho^{d+1}(\alpha) = \varphi\rho^d(\alpha)0$$

Lemma 5 *Let d be a natural number.*

(1) $\mathcal{B} = \bigcup\{\mathcal{B}^d \mid d \in \omega\}$.
(2) *If* $\alpha \in \mathcal{B}^d$, *then* $\alpha < \rho^{d+1}(o)$ *and* $\rho^k(\alpha) \in \mathcal{B}^{d+k}$.
(3) $\rho^{d+1}(o) \in \mathcal{B}^{d+1} \setminus \mathcal{B}^d$.
(4) *If* $\alpha < \beta$, *then* $\rho^d(\alpha) < \rho^d(\beta)$.
(5) *If* $\alpha \trianglelefteq \beta$, *then* $\rho^d(\alpha) \trianglelefteq \rho^d(\beta)$.
(6) *If* $\alpha \in \mathcal{B}^{d+1} \setminus \mathcal{B}^d$ *and* $\beta \in \mathcal{B}^d$, *then* $\alpha \ntrianglelefteq \beta$ *and* $\beta < \alpha$.

PROOF. The first five claims are obvious. We show the last assertion by induction on α and β. If $\beta = 0$ there is nothing to show. Let $\alpha = \varphi\alpha_1\alpha_2$ and $\beta = \varphi\beta_1\beta_2$. If $\alpha_1 \in \mathcal{B}^{d-1}$, then $\alpha_2 \in \mathcal{B}^{d+1} \setminus \mathcal{B}^d$. Hence $\beta < \alpha_2 < \alpha$ by induction hypothesis. Now assume $\alpha_1 \in \mathcal{B}^d \setminus \mathcal{B}^{d-1}$. Then $\beta_1 < \alpha_1$ and $\beta_2 < \alpha$ by induction hypothesis., so $\beta < \alpha$ and $\alpha \ntrianglelefteq \beta$. □

Note that ω and B^1 can be identified by the isomorphism f defined as follows: $f(0) := o$ and $f(n+1) := \varphi(o, f(n))$. Hence we may talk about occurrences of natural numbers in $\alpha \in \mathcal{B}^d$ for $d \geq 1$.

For $k \geq 1$ define

- $\mathcal{B}^{1,k} := \{0, 1, \ldots, k-1\}$.
- $\mathcal{B}^{d+1,k} := \{\alpha \mid \alpha = 0 \text{ or } \alpha = \varphi\beta\gamma, \text{ where } \beta \in \mathcal{B}^{d,k} \text{ and } \gamma \in \mathcal{B}^{d+1,k}\}$.

Lemma 6 *Let d, k be natural numbers.*

(1) $\mathcal{B}^d = \bigcup_{k>0} \mathcal{B}^{d,k}$.
(2) *If* $\alpha \in \mathcal{B}^{d+1,k}$, *then* $\alpha < \rho^d(k)$.
(3) *If* $\alpha \in \mathcal{B}^{d,k+1} \setminus \mathcal{B}^{d,k}$ *and* $\beta \in \mathcal{B}^{d,k}$, *then* $\alpha \ntrianglelefteq \beta$ *and* $\beta < \alpha$.

PROOF. Similar to Lemma 5. Every claim can be shown by an simple induction on k. $\square$

Given a positive natural number n define $\mathcal{B}_n$ by

$$\mathcal{B}_n := \begin{cases} \mathcal{B}^{d+1} & \text{if } n = 2d \\ \mathcal{B}^{d+1,2} & \text{if } n = 2d-1 \,. \end{cases}$$

We claim

$$o(\mathcal{B}_n, \trianglelefteq \restriction \mathcal{B}_n) = otyp(< \restriction \mathcal{B}_n) = \omega_{n+1} \,.$$

3 Maximal order types

It is in general not a simple task to decide the maximal order type of a *wpo*. Some interesting methods are introduced in [7,3,13]. However, there is a problem that in most cases they can be carried out only in a long-winded way. Fortunately, there is a much more simple way for our case. We are going to take a well-known *wpo* and compare it with $(\mathcal{B}_n, \trianglelefteq)$.

Note first that $\mathcal{B}^{d+1}$ and $(\mathcal{B}^d)^*$ are similarly constructed. Every $\alpha \in \mathcal{B}^{d+1}$ is of the form $\alpha = \varphi\alpha_1\varphi\alpha_2\cdots\varphi\alpha_m o$, where $\alpha_i \in \mathcal{B}^d$. If $\beta = \varphi\beta_1\varphi\beta_2\cdots\varphi\beta_n o \in \mathcal{B}^{d+1}$ and $\alpha_1\cdots\alpha_m \trianglelefteq_{\mathrm{H}} \beta_1\cdots\beta_n$ then $\alpha \trianglelefteq \beta$. Though this relationship is not isomorphic, we can indeed show that $o(\mathcal{B}^{d+1}, \trianglelefteq) = o((\mathcal{B}^d)^*, \trianglelefteq_{\mathrm{H}})$.

We need the following obvious fact.

Lemma 7 *Let $(A, \preceq_1)$ and $(B, \preceq_2)$ be wpo's and $f\colon A \to B$ an injective function such that*

$$a \preceq_1 b \iff f(a) \preceq_2 f(b)$$

for all a, $b \in A$. Then it holds that $o(A) \le o(B)$.

Theorem 8 *For any $d > 0$, $o(\mathcal{B}^{d+1}, \trianglelefteq) = o(\mathcal{B}^{d+1} \setminus \mathcal{B}^d, \trianglelefteq) = o((\mathcal{B}^d)^*, \trianglelefteq_{\mathrm{H}})$.*

PROOF. Define $f\colon \mathcal{B}^{d+1} \to (\mathcal{B}^d)^*$ and $g\colon (\mathcal{B}^d)^* \to \mathcal{B}^{d+1} \setminus \mathcal{B}^d$ defined as follows:

$$f(\alpha) := \begin{cases} \epsilon & \text{if } \alpha = o \\ \alpha & \text{if } \alpha = \varphi\alpha_1\alpha_2 \in \mathcal{B}^d \\ \alpha_1, f(\alpha_2) & \text{if } \alpha = \varphi\alpha_1\alpha_2 \notin \mathcal{B}^d \end{cases}$$

and

$$g(\alpha_1, \ldots, \alpha_m) := \varphi\alpha_1\varphi\alpha_2\cdots\varphi\alpha_m\rho^{d+1}(o)$$

where ϵ denotes the empty sequence. It is then very easy to show that f and g satisfy the conditions in Lemma 7. In case of the function f, one proves it by induction on the length of its argument. Here we demonstrate one critical case: $\alpha = \varphi\alpha_1\alpha_2$, $\beta = \varphi\beta_1\beta_2 \in \mathcal{B}^{d+1} \setminus \mathcal{B}^d$ and $\alpha \trianglelefteq \beta_2$. Then $\beta_2 \in \mathcal{B}^{d+1} \setminus \mathcal{B}^d$ and by induction hypothesis $\alpha_1, f(\alpha_2) \trianglelefteq_{\mathrm{H}} f(\beta_2)$. Hence it follows that $\alpha_1, f(\alpha_2) \trianglelefteq_{\mathrm{H}} \beta_1, f(\beta_2)$. The rest can be handled similarly. $\square$

Corollary 9 *For any $d > 0$, $(\mathcal{B}^d, \trianglelefteq)$ is a wpo and $o(\mathcal{B}^d, \trianglelefteq) = \omega_{2d-1}$.*

PROOF. By induction on $d > 0$. If $d = 1$, then $\mathcal{B}^1 = \{o, \varphi oo, \varphi o(\varphi oo), \dots\}$ is linearly ordered by $\trianglelefteq$ and so $o(\mathcal{B}^1, \trianglelefteq) = \omega$. If $d > 1$, use induction hypothesis, Theorem 8, and Theorem 4. $\square$

Corollary 10 *$(\mathcal{B}, \trianglelefteq)$ is a well-ordering and $o(\mathcal{B}) = \varepsilon_0$.*

Lemma 11 *Let d, k be positive natural numbers. Then*

$$o(\mathcal{B}^{d,k}, \trianglelefteq \restriction \mathcal{B}^{d,k}) = \begin{cases} k & \text{if } d = 1 \\ \omega_{2(d-1)}(k-1) & \text{otherwise}. \end{cases}$$

PROOF. Similar to Corollary 9 $\square$.

Theorem 12 *$o(\mathcal{B}_n, \trianglelefteq \restriction \mathcal{B}_n) = \omega_{n+1}$ for any positive natural number n.*

PROOF. It follows directly from Corollary 9 and Lemma 11 for $k = 2$. $\square$

4 Order types

We are now going to compute the order types of $(\mathcal{B}_n, < \restriction \mathcal{B}_n)$. It is not so obvious as it might seem. $\mathcal{B}$ will be considered as ordinal notation systems based on the recursive path ordering on strings.

Definition 13 (Recursive path ordering) *Let $(\mathcal{A}, \prec)$ be a well-ordering. The* recursive path ordering *$\prec_{rpo}$ on $\mathcal{A}^*$ is defined as follows: Let ϵ be the empty list.*

- *If $\epsilon \prec_{rpo} u$ for $u \neq \epsilon$.*
- *If $u = au_1$ and $v = bv_1$, then $u \prec_{rpo} v$ if one of the following holds:*

(1) $a \prec b$ and $u_1 \prec_{rpo} v$;
(2) $a = b$ and $u_1 \prec_{rpo} v_1$;

(3) $b \prec a$ *and* $u \preceq_{rpo} v_1$.

Dershowitz [5] shows that the recursive path ordering preserves the well-orderedness.

Theorem 14 (Dershowitz) *If* $(\mathcal{A}, \prec)$ *is a well-ordering, so is* $(\mathcal{A}^*, \prec_{rpo})$.

Let $\xi < \varepsilon_0$ be the order type of $\prec$ on $\mathcal{A}$ and $\eta \mapsto a_\eta$, $\eta < \xi$, the enumeration function of $\mathcal{A}$. Using the idea elaborated by Touzet [19], we are going to decide the order type of $\prec_{rpo}$ on $\mathcal{A}^*$.

Lemma 15 *For each limit ordinal* $\alpha < \omega^{\omega^\xi}$, *there are uniquely determined* γ, β, *and* $\eta < \xi$ *such that*

(1) $\alpha = \gamma + \omega^{\omega^\eta} \cdot \beta$,
(2) $0 < \beta < \omega^{\omega^{\eta+1}}$, *and*
(3) there are no $\mu \in \omega^{\omega^{\eta+1}} \setminus \{0\}$ *and* $\delta \in \omega^{\omega^\xi}$ *such that* $\gamma = \delta + \mu$.

PROOF. Let $\alpha =_{NF} \omega^{\alpha_0} + \cdots + \omega^{\alpha_n}$. Let $\eta < \xi$ and j be such that

$$\omega^\eta \leq \alpha_n < \omega^{\eta+1} \quad \text{and} \quad j := \min\{k \mid \omega^\eta \leq \alpha_k < \omega^{\eta+1}\}\,.$$

There are δ_k, $j \leq k \leq n$, such that $\alpha_j = \omega^\eta + \delta_j, \ldots, \alpha_n = \omega^\eta + \delta_n$. Hence it holds that $\alpha = \gamma + \omega^{\omega^\eta} \cdot \beta$, where $\gamma =_{NF} \omega^{\alpha_0} + \cdots + \omega^{\alpha_{j-1}}$ and $\beta =_{NF} \omega^{\delta_j} + \cdots + \omega^{\delta_n}$, and η, β, γ satisfy the conditions (2) and (3).

We now prove the uniqueness of the decomposition. Let η', β', γ' also satisfy (1) $\sim$ (3). If $\beta =_{NF} \omega^{\beta_0} + \cdots + \omega^{\beta_m}$ and $\beta' =_{NF} \omega^{\beta'_0} + \cdots + \omega^{\beta'_\ell}$ and if γ is in Cantor normal form too, then conditions (2) and (3) guarantee that

$$\alpha =_{NF} \gamma + \omega^{\omega^\eta + \beta_1} + \cdots + \omega^{\omega^\eta + \beta_m} =_{NF} \gamma' + \omega^{\omega^{\eta'} + \beta'_1} + \cdots + \omega^{\omega^{\eta'} + \beta'_\ell}$$

and hence $\eta = \eta'$. Suppose for instance $\gamma < \gamma'$. Then $\gamma' = \gamma + \omega^{\omega^\eta + \beta_1} + \cdots + \omega^{\omega^\eta + \beta_p}$ for some $p \leq m$. This contradicts (3). So $\gamma = \gamma'$ and hence $m = \ell$, $\beta_k = \beta'_k$, $1 \leq k \leq m$. $\square$

In the sequel, $\gamma + \omega^{\omega^\eta} \cdot \beta$ means always in the sense of Lemma 15. For ordinals $\beta > 0$, $-1 + \beta$ denotes $\beta - 1$ if $\beta < \omega$ and β otherwise.

Definition 16 *Let* $(\mathcal{A}, \prec)$ *be a well-ordering and otyp*$(\prec) = \xi \in \varepsilon_0 \setminus \{0\}$. *The function* $\mathcal{O} \colon \omega^{\omega^{-1+\xi}} \to \mathcal{A}^*$ *is defined by:*

$$\mathcal{O}(\alpha) := \begin{cases} \epsilon & \text{if } \alpha = 0 \\ a_0 \mathcal{O}(\beta) & \text{if } \alpha = \beta + 1 \\ a_{1+\eta} \mathcal{O}(-1+\beta) \mathcal{O}(\gamma) & \text{if } \alpha = \gamma + \omega^{\omega^\eta} \cdot \beta\,. \end{cases}$$

Now we are going to show that the definition of $((B^d)^*, <_{rpo})$ is just another way to see $(B^{d+1}, <)$.

Theorem 17 *Let $(\mathcal{A}, \prec)$ be a well-ordering. If $otyp(\prec) = \xi \in \varepsilon_0 \setminus \{0\}$ on $\mathcal{A}$, then we have on $\mathcal{A}^*$*

$$otyp(\prec_{rpo}) = \omega^{\omega^{-1+\xi}} = \begin{cases} \omega^{\omega^{\xi-1}} & \text{if } \xi \in \omega \setminus \{0\} \\ \omega^{\omega^{\xi}} & \text{otherwise}. \end{cases}$$

PROOF. We show that the function $\mathcal{O}\colon (\omega^{\omega^{-1+\xi}}, <) \to (\mathcal{A}^*, \prec_{rpo})$ is an isomorphism.

(1) $\mathcal{O}$ is order-preserving, i.e. $\mathcal{O}(\alpha) \prec_{rpo} \mathcal{O}(\beta)$ if $\alpha < \beta$. Note that the ordering $<$ on ordinals is the transitive closure of the schemes $\forall n \in \omega(\alpha_n < \alpha)$, where $(\alpha_n)_n$ builds a fundamental sequence for α. (The definition of the fundamental sequence will be directly given below during the proof.) So it suffices to show that $\forall n \in \omega(\mathcal{O}(\alpha_n) \prec_{rpo} \mathcal{O}(\alpha))$ for any $\alpha < \xi$.

(a) $\alpha = \beta + 1$: Then $\alpha_n = \beta$ and $\mathcal{O}(\alpha_n) = \mathcal{O}(\beta) \prec_{rpo} a_0\mathcal{O}(\beta) = \mathcal{O}(\alpha)$.

(b) $\alpha = \gamma + \omega^{\omega^\eta} \cdot (\beta + 1)$:

- $\eta = 0$, i.e. $\alpha_n = \gamma + \omega^{\omega^0} \cdot \beta + n + 1$: Then

$$\mathcal{O}(\alpha_n) = \begin{cases} a_0^{n+1}\mathcal{O}(\gamma) & \text{if } \beta = 0 \\ a_0^{n+1}a_1\mathcal{O}(-1+\beta)\mathcal{O}(\gamma) & \text{otherwise} \end{cases}$$

$$\prec_{rpo}$$

$$\mathcal{O}(\alpha) = \begin{cases} a_1\mathcal{O}(\gamma) & \text{if } \beta = 0 \\ a_1\mathcal{O}(-1+\beta+1)\mathcal{O}(\gamma) & \text{otherwise}. \end{cases}$$

- $\eta = \eta_0 + 1$, i.e. $\alpha_n = \gamma + \omega^{\omega^\eta} \cdot \beta + \omega^{\omega^{\eta_0}} \cdot \omega^{\omega^{\eta_0} \cdot n}$: Then

$$\mathcal{O}(\alpha_n) = \begin{cases} a_{1+\eta_0}^{n+1}\mathcal{O}(\gamma) & \text{if } \beta = 0 \\ a_{1+\eta_0}^{n+1}a_{1+\eta}\mathcal{O}(-1+\beta)\mathcal{O}(\gamma) & \text{otherwise} \end{cases}$$

$$\prec_{rpo}$$

$$\mathcal{O}(\alpha) = \begin{cases} a_{1+\eta}\mathcal{O}(\gamma) & \text{if } \beta = 0 \\ a_{1+\eta}\mathcal{O}(-1+\beta+1)\mathcal{O}(\gamma) & \text{otherwise}. \end{cases}$$

- η is a limit ordinal, i.e.$\alpha_n = \gamma + \omega^{\omega^\eta} \cdot \beta + \omega^{\omega^{\eta_n}}$: Then

$$\mathcal{O}(\alpha_n) = \begin{cases} a_{1+\eta_n}\mathcal{O}(\gamma) & \text{if } \beta = 0 \\ a_{1+\eta_n}a_\eta\mathcal{O}(-1+\beta)\mathcal{O}(\gamma) & \text{otherwise} \end{cases}$$

$$\prec_{rpo}$$

$$\mathcal{O}(\alpha) = \begin{cases} a_\eta\mathcal{O}(\gamma) & \text{if } \beta = 0 \\ a_\eta\mathcal{O}(-1+\beta+1)\mathcal{O}(\gamma) & \text{otherwise}. \end{cases}$$

(c) $\alpha = \gamma + \omega^{\omega^\eta} \cdot \lambda$, where λ is a limit ordinal: Then $\alpha_n = \gamma + \omega^{\omega^\eta} \cdot \lambda_n$ and $\mathcal{O}(\alpha_n) = a_{1+\eta}\mathcal{O}(-1+\lambda_n)\mathcal{O}(\gamma) \prec_{rpo} a_{1+\eta}\mathcal{O}(-1+\lambda)\mathcal{O}(\gamma) = \mathcal{O}(\alpha)$.

We have shown that $\mathcal{O}$ is order-preserving, so it is injective.

(2) Let $u \in \mathcal{A}^*$. By induction on the length of u we show that there is an $\alpha < \omega^{\omega^\xi}$ such that $\mathcal{O}(\alpha) = u$.

(a) $u = \epsilon$: $\mathcal{O}(0) = \epsilon$.

(b) $u = a_0 v$: Then $\mathcal{O}(\beta + 1) = a_0 v$, where $\mathcal{O}(\beta) = v$.

(c) $u = a_\eta v$, $\eta > 0$: Then let $\eta' = \eta$ if $\eta \geq \omega$ and $\eta' = \eta + 1$ otherwise.

- $v \in \{a_0, \ldots, a_\eta\}^*$: Let $\mathcal{O}(-1+\beta) = v$. Then $-1+\beta < \omega^{\omega^{\eta'}}$ and $\mathcal{O}(\omega^{\omega^{-1+\eta}} \cdot \beta) = a_\eta\mathcal{O}(-1+\beta) = a_\eta v = u$.

  Note that this case implies, in particular, that $\mathcal{O}\colon \omega^{\omega^{\xi-1}} \to \mathcal{A}^*$ is an isomorphism if $\xi \in \omega \setminus \{0\}$. Indeed, if $\mathcal{A} = \{a_0, \ldots, a_\eta\}$ and $\xi = \eta + 1$ then we have just shown that $\alpha < \omega^{\omega^\eta}$ for α such that $\mathcal{O}(\alpha) = u$.
- $v \notin \{a_0, \ldots, a_\eta\}^*$: Let $b \in \mathcal{A} \setminus \{a_0, \ldots, a_\eta\}$, $v_1 \in \{a_0, \ldots, a_\eta\}^*$, and $v_2 \in \mathcal{A}^*$ such that $v = v_1 b v_2$. Let $\mathcal{O}(-1+\beta) = v_1$ and $\mathcal{O}(\gamma) = b v_2$. Then $\mathcal{O}(\gamma + \omega^{\omega^{-1+\eta}} \cdot \beta) = a_\eta\mathcal{O}(-1+\beta)\mathcal{O}(\gamma) = a_\eta v_1 b v_2 = u$. □

Corollary 18 *For any $d > 0$, $(\mathcal{B}^d, <)$ is a well-ordering and otyp$(< \restriction \mathcal{B}^d) = \omega_{2d-1}$.*

PROOF. Note just that $(\mathcal{B}^{d+1}, <)$ is isomorphic to $((\mathcal{B}^d)^*, <_{rpo})$. □

Corollary 19 *$(\mathcal{B}, <)$ is a well-ordering and otyp$(<) = \varepsilon_0$.*

Lemma 20 *Let d, k be positive natural numbers. Then*

$$\mathit{otyp}(< \restriction \mathcal{B}^{d,k}) = \begin{cases} k & \text{if } d = 1 \\ \omega_{2(d-1)}(k-1) & \text{otherwise}. \end{cases}$$

PROOF. Similar to Corollary 18. □

Theorem 21 *otyp$(\mathcal{B}_n, < \restriction \mathcal{B}_n) = \omega_{n+1}$ for any positive natural number n.*

PROOF. It follows directly from Corollary 18 and Lemma 21 for $k = 2$. $\square$

Finally, Theorem 12 and Theorem 21 imply the main claim.

Theorem 22 *$o(\mathcal{B}_n, \trianglelefteq \restriction \mathcal{B}_n) = otyp(< \restriction \mathcal{B}_n) = \omega_{n+1}$ for any positive natural number n.*

5 Characterization of $\omega_n(k)$ by binary trees

Similarly to the case of finite trees, one can define the slowly-well-orderedness of binary trees, $\mathrm{SWP}(\mathcal{B}, \preceq, f)$. This is a true Π^0_2 sentence. Weiermann showed, however, that its PA-provability depends on f:

Theorem 23 (Weiermann [18]) *Given a primitively recursive real number r, set $f_r(i) := r \cdot \log_2(i)$. Then $\mathrm{SWP}(\mathcal{B}, \preceq, f_r)$ is PA-provable iff $r \leq \frac{1}{2}$.*

As for the proof, he used the cumulative hierarchy $(\mathcal{B}^d)_d$ which made it possible to approach $\mathcal{B}$ from down. Using this idea of approach from down, one can obtain similar results with respect to $\mathcal{B}_n$. Let $\mathrm{I}\Sigma_n$ be the fragment of PA where the induction scheme is restricted to Π^0_n formulae.

Theorem 24 (Lee and Weiermann [20]) *There exists a sequence $(r_n)_{n\geq 1}$ of real numbers such that for any primitively recursive real number r it holds that:*

(1) $r_{n+1} < r_n$ *and* $\lim_{n\to\infty} r_n = \frac{1}{2}$.
(2) $\mathrm{SWP}(\mathcal{B}_n, \trianglelefteq, f_r)$ *is* $\mathrm{I}\Sigma_n$*-provable iff* $r \leq r_n$.

For the proof we had to define other cumulative hierarchies. In case of $\mathcal{B}_n = \mathcal{B}^d$ for some d, we can use the hierarchies $(\mathcal{B}^{d,k})_k$ already defined. However different hierarchies are needed when $\mathcal{B}_n = \mathcal{B}^{d,2}$ for some $d \geq 2$. Define the sets $\mathcal{S}^{d,k}$, $k \in \mathbb{N}$, recursively:

- $\mathcal{S}^{2,k} := \{\alpha \in \mathcal{B}^{2,2} \mid 1 \text{ occurs in } \alpha \text{ at most } k \text{ times}\}$.
- $\mathcal{S}^{d+1,k} := \{\alpha \mid \alpha = \varphi\beta\gamma \text{ for some } \beta \in \mathcal{S}^{d,k} \text{ and } \gamma \in \mathcal{S}^{d+1,k}\} \cup \{o\}$.

We use the following notation: $\varphi^0_o\gamma = \gamma$ and $\varphi^{m+1}_o\gamma = \varphi o(\varphi^m_o\gamma)$.

Lemma 25 *Let $d \geq 2$.*

(1) $\mathcal{S}^{d,k} \subseteq \mathcal{B}^{d,2}$ *and* $\mathcal{B}^{d,2} = \bigcup_{k\in\mathbb{N}} \mathcal{S}^{d,k}$.
(2) *If* $\alpha \in \mathcal{S}^{d,k+1} \setminus \mathcal{S}^{d,k}$ *and* $\beta \in \mathcal{S}^{d,k}$*, then* $\beta < \alpha$ *and* $\alpha \ntrianglelefteq \beta$.
(3) $o(\mathcal{S}^{d,k}, \trianglelefteq \restriction \mathcal{S}^{d,k}) = otyp(< \restriction \mathcal{S}^{d,k}) = \omega_{2d-3}(k+1)$.

PROOF. (1) is clearly true. The second assertion can be handled similarly to Lemma 5. By induction on d, we show (3). Let $d = 2$. The case $k = 0$ is obvious. Assume now $k = \ell + 1$.

- $otyp(< \restriction \mathcal{S}^{2,\ell+1})$: Note first that

$$\alpha \in \mathcal{S}^{2,\ell+1} \iff \begin{cases} \alpha = o\,, \\ \exists\beta\,(\beta \in \mathcal{S}^{2,\ell+1} \text{ and } \alpha = \varphi o \beta)\,, \text{ or} \\ \exists\beta\,(\beta \in \mathcal{S}^{2,\ell} \text{ and } \alpha = \varphi 1 \beta)\,. \end{cases}$$

  Hence for any $\alpha \in \mathcal{S}^{2,\ell+1} \setminus \mathcal{S}^{2,\ell}$ there are $\beta \in \mathcal{S}^{2,\ell} \setminus \mathcal{S}^{2,\ell-1}$ (or $\beta \in \mathcal{S}^{2,\ell}$ if $\ell = 0$) and $m \in \omega$ such that $\alpha = \varphi_o^m \varphi 1 \beta$. Now it is easy to see that

$$otyp(< \restriction \mathcal{S}^{2,\ell+1}) = \omega^{\ell+1} + \omega^{\ell+2} = \omega^{\ell+2}$$

  since we know (2) and $otyp(< \restriction \mathcal{S}^{2,\ell}) = \omega^{\ell+1}$ by induction hypothesis.

- $o(\mathcal{S}^{2,\ell+1}, \trianglelefteq \restriction \mathcal{S}^{2,\ell+1})$: Assume $\ell > 0$. The case $\ell = 0$ can be handled similarly. Given β, $\gamma \in \mathcal{S}^{2,\ell} \setminus \mathcal{S}^{2,\ell-1}$ we first assert that

$$\varphi_0^m \varphi 1 \beta \trianglelefteq \varphi_0^n \varphi 1 \gamma \ \text{ iff } \ (m \leq n \,\text{and}\, \beta \trianglelefteq \gamma)\,.$$

  If $m \leq n$ and $\beta \trianglelefteq \gamma$ we obviously have $\varphi_0^m \varphi 1 \beta \trianglelefteq \varphi_0^n \varphi 1 \gamma$. If $m > n$ one can show by induction on n that $\varphi_0^m \varphi 1 \beta \ntrianglelefteq \varphi_0^n \varphi 1 \gamma$. Now assume $m \leq n$ and $\varphi_0^m \varphi 1 \beta \trianglelefteq \varphi_0^n \varphi 1 \gamma$. Then we should have $\varphi 1 \beta \trianglelefteq \varphi 1 \gamma$ and hence $\beta \trianglelefteq \gamma$.

  However, this assertion implies that we may consider $\mathcal{S}^{2,\ell+1} \setminus \mathcal{S}^{2,\ell}$ as the Cartesian product of $(\mathbb{N}, \leq)$ and $(\mathcal{S}^{2,\ell} \setminus \mathcal{S}^{2,\ell-1}, \trianglelefteq)$, where $(\mathbb{N}, \leq)$ is the standard less-than-or-equal-to relation on natural numbers:

$$(\mathcal{S}^{2,\ell+1} \setminus \mathcal{S}^{2,\ell}, \trianglelefteq) \cong (\mathbb{N}, \leq) \times (\mathcal{S}^{2,\ell} \setminus \mathcal{S}^{2,\ell-1}, \trianglelefteq)$$

  Using Lemma 2, induction hypothesis, and the idea used in the proof of Theorem 8, it holds that

$$\begin{aligned} o(\mathcal{S}^{2,\ell+1}, \trianglelefteq) &= o(\mathcal{S}^{2,\ell+1} \setminus \mathcal{S}^{2,\ell}, \trianglelefteq) \\ &= o(\mathbb{N}, \leq) \otimes o(\mathcal{S}^{2,\ell} \setminus \mathcal{S}^{2,\ell-1}, \trianglelefteq) \\ &= \omega \otimes o(\mathcal{S}^{2,\ell}, \trianglelefteq) \; = \; \omega^{\ell+2}\,. \end{aligned}$$

For $d > 2$, we can proceed as before using Higman's lemma (Theorem 4) and recursive path ordering (Theorem 17), respectively. □

Finally, we characterize $\omega_n(k)$ using binary trees. Given positive natural numbers n and k, define $\mathcal{B}_{n,k}$ by

$$\mathcal{B}_{n,k} := \begin{cases} \mathcal{B}^{d,k+1} & \text{if } n = 2d-2 \\ \mathcal{S}^{d,k-1} & \text{if } n = 2d-3\,. \end{cases}$$

Theorem 26 *For all positive natural numbers n and k, we have that*

$$o(\mathcal{B}_{n,k}, \trianglelefteq \restriction \mathcal{B}_{n,k}) = otyp(< \restriction \mathcal{B}_{n,k}) = \omega_n(k)\,.$$

PROOF. The claim follows directly from Lemma 20 and Lemma 25. □

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