A comparison of well-known ordinal notation systems for ε_0

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Abstract

We consider five ordinal notation systems of ε_0 which are all well-known and of interest in proof-theoretic analysis of Peano arithmetic: Cantor's system, systems based on binary trees and on countable tree-ordinals, and the systems due to Schütte-Simpson and Beklemishev.

The main point of this paper is to demonstrate that the systems except the system based on binary trees are equivalent as structured systems, in spite of the fact that they have their origins in different views and trials in proof theory. This is true while Weiermann's results based on Friedman-style miniaturization indicate that the system based on binary trees is of different character than the others.

Key words: Peano arithmetic, ordinal notation systems, Friedman-style miniaturization, slowly-well-orderedness, phase transition
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1 Introduction

Starting with Gentzen's consistency proof for Peano arithmetic (PA) in [1], a number of primitive recursive ordinal notation systems have been defined to give consistency proofs for PA, each of them provided with a primitive recursive well-ordering.

Most famous one is probably Cantor's system of the ordinals in normal form less than ε_0 with a natural well-ordering. Using a transfinite induction along the ordering, the consistency of PA can be shown while, for any proper initial

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segment of the ordering, it is PA-provable that the transfinite induction on the segment can be applied to arbitrary formulas. This kind of characterization of PA is true for all of its well-known ordinal number systems.

Another popular characterization of a theory is formulated by the question whether there is a natural ordinal classification of all provably total functions of the theory. In a significant number of well-known theories in proof theory there do exist plausible solutions, such as the classifications of provably recursive functions through *Hardy-Wainer* or *Schwichtenberg-Wainer hierarchies.*

However, there still remain many questions about ordinal notation systems of a theory: How is it to be explained that proof-theoretic ordinals are sensitive to the choice of particular proof systems? What are the intrinsic properties that distinguish some ordering from the others? In proof theory, in fact, it is one of the so-called conceptual problems to give criteria for *natural* or *canonical* ordinal notations. See [2–4] for intensive discussions.

One of the latest approaches to this problem is done by Beklemishev [5]. He was concerned with the question how to recover an ordinal notation system *from* a given theory. He posed the question what would make a formal theory possible to rigorously specify its canonical ordinal notation system? He pointed out that an algebraic view point of proof theory, e.g., a well-behaved notion of *graded provability algebra*, could give an answer. He showed this for PA in a canonical way.

Beside these two systems, Cantor's and Beklemishev's systems, three further ordinal notation systems for ε_0 are taken into account in this paper:

- *System based on rooted binary trees*: A rooted binary tree is a set of nodes such that, if it is not empty, there is one distinguished node called the root and the remaining nodes are partitioned into two rooted binary trees. The homeomorphic embeddability relation $\trianglelefteq$ on the set $\mathcal{B}$ of all rooted binary trees is well-founded and has the maximal order type ε_0. The system $(\mathcal{B}, <)$ can also be obtained from the Feferman-Schütte notation system for Γ_0 by omitting the addition operator, where $<$ is a canonical extension of $\trianglelefteq$.
- *System based on countable tree-ordinals*: A certain set of countable tree-ordinals resembles the Cantor system. The sub-tree ordering on the set builds no well-ordering, although it is well-founded and has the height ε_0.
- *Schütte-Simpson's system*: The system is obtained from Buchholz's ordinal notation system in [6] by omitting the addition and the construction of ω^α as basic operators. This system contains a subset with a well-ordering of order type ε_0.

We will show that the four systems except the system based on binary trees build equivalent *structured systems* though they have all different historical backgrounds in proof theory. The equivalence between any two stuctured system will be established by a relatively simple construction of order-preserving isomorphism. See Section 3.

A *structured system* of countable ordinals is a system where an arbitrary, but fixed *fundamental sequence* has been assigned to each limit. In case of Cantor's system, e.g., the fundamental sequence for any limit $\lambda < \varepsilon_0$ can be defined as follows: Let $\lambda = \omega^{\lambda_1} + \cdots + \omega^{\lambda_k}$ be in Cantor normal form.

$$\lambda[n] := \begin{cases} \omega^{\lambda_1} + \cdots + \omega^{\lambda_{k-1}} + \omega^{\lambda_k[n]} & \text{if } \lambda_k \text{ is a limit,} \\ \omega^{\lambda_1} + \cdots + \omega^{\lambda_{k-1}} + \omega^{\lambda_k - 1} \cdot (n+1) & \text{otherwise}. \end{cases}$$

Then $\lambda[n] < \lambda[n+1]$ and $\lim_{n\to\infty} \lambda[n] = \lambda$.

Given $f\colon \mathbb{N} \to \mathbb{N}$, we define the iterations: $f^{(0)}(i) := i$ and $f^{(\ell+1)}(i) := f(f^{(\ell)}(i))$. Then the Hardy-Wainer hierarchy $(H_\alpha)_{\alpha<\varepsilon_0}$ and the Schwichtenberg-Wainer hierarchy $(F_\alpha)_{\lambda<\varepsilon_0}$ are defined as follows:

$$\begin{aligned} H_0(i) &= i & \qquad\qquad F_0(i) &= i+1 \\ H_{\alpha+1}(i) &= H_\alpha(i+1) \quad \text{and} & F_{\alpha+1}(i) &= F_\alpha^{(i+1)}(i) \\ H_\lambda(i) &= H_{\lambda[n]}(i) & F_\lambda(i) &= F_{\lambda[n]}(i) \end{aligned}$$

Further let $H_{\varepsilon_0}(i) := H_{\omega_i}(i)$ and $F_{\varepsilon_0}(i) := F_{\omega_i}(i)$. Then $F_\alpha(i) = H_{\omega^\alpha}(i)$. And it is a folklore in proof theory that H_α (resp. F_α) is provably recursive in PA iff $\alpha < \varepsilon_0$. See [7] or [8] for details.

Friedman-style miniaturization and phase transition *Phase transition* in physics means the transformation of a thermodynamic system from one phase to another. The distinguishing characteristic of phase transition is an abrupt change in physical properties with a small change in a thermodynamic variable such as the temperature.

It was the paper [9] by Loebl and Matoušek on Friedman-style miniaturization of Kruskal's theorem which indicated that similar phenomenon could happen in proof theory: abrupt change of provability of a sentence with a small change of a parameter. And it is one of the starting points of Weiermann's pioneer works on phase transition.

A *finite rooted tree* is a finite partial ordering $(T, \preceq)$ such that, if T is not empty,

- T has a smallest element called the *root* of T;
- for each $b \in T$, the set $\{a \in T\colon a \preceq b\}$ is totally ordered.

Let $a \wedge b$ denote the infimum of a and b for a, $b \in T$. Given finite rooted trees

T_1 and T_2, a *homeomorphic embedding* of T_1 into T_2 is an one-to-one mapping $f\colon T_1 \to T_2$ such that $f(a \wedge b) = f(a) \wedge f(b)$ for all $a, b \in T_1$. We write $T_1 \trianglelefteq T_2$ if there exists a homeomorphic embedding $f\colon T_1 \to T_2$. The following true Π^1_1-sentence says that $\langle \mathbb{T}, \trianglelefteq \rangle$ is a *well-partial ordering*, where $\mathbb{T}$ is the set of all finite rooted trees, cf. Kruskal [10].

Theorem 1 (Kruskal's Theorem) *For any infinite sequence of finite rooted trees $(T_k)_{k<\omega}$, there are indices $\ell < m$ satisfying $T_\ell \trianglelefteq T_m$.*

Let $\|T\|$ denote the number of nodes of the finite tree T. Assume further that the set of finite rooted trees is coded primitive recursively into a set of natural numbers as usual. Given $f\colon \mathbb{N} \to \mathbb{N}$, the *slowly well-partially-orderedness* is defined as follows:

> For any k there exists a constant n so large that, for any finite sequence $T_0, \ldots, T_n$ of finite rooted trees with $\|T_i\| \leq k + f(i)$ for all $i \leq n$, there are indices $\ell < m \leq n$ satisfying $T_\ell \trianglelefteq T_m$.

Let $\mathrm{SWP}(\mathbb{T}, \trianglelefteq, f)$ be the above Π^0_2-sentence.

Theorem 2 (Friedman [11], Smith [12]) $\mathrm{ATR}_0 \nvdash \mathrm{SWP}(\mathbb{T}, \trianglelefteq, id)$.

Theorem 3 (Loebl and Matoušek [9]) *Let $f_r(i) := r \cdot |i|$. Then*

$$\mathrm{PA} \nvdash \mathrm{SWP}(\mathbb{T}, \trianglelefteq, f_4) \quad \textit{and} \quad \mathrm{PA} \vdash \mathrm{SWP}(\mathbb{T}, \trianglelefteq, f_{\frac{1}{2}}).$$

These very interesting results made one speculate that there would be a threshold in view of the PA-provability of $\mathrm{SWP}(\mathbb{T}, \trianglelefteq, f_r)$ depending on the numbers between 1/2 and 4. Indeed, Weiermann found such a point which is closely related to the so-called *Otter's tree constant* $\alpha = 2.955765\ldots$:

$$t(n) \sim \beta \cdot \alpha^n \cdot n^{-\frac{2}{3}}$$

for some real number β, where $t(n) = card(\{T\colon \|T\| = n\})$, cf. Otter [13].

Theorem 4 (Weiermann [14]) *Let $c = \frac{1}{\log(\alpha)}$ and r be a primitively recursive real number. Set $f_r(i) := r \cdot |i|$. Then*

$$\mathrm{PA} \vdash \mathrm{SWP}(\mathbb{T}, \trianglelefteq, f_r) \quad \textit{iff} \quad r > c.$$

In fact, one can even show that the same phase transition holds with respect to $\mathrm{ACA}_0 + \Pi^1_2\text{-BI}$, cf. Lee [15]:

$$\mathrm{ACA}_0 + \Pi^1_2\text{-BI} \nvdash \mathrm{SWP}(\mathbb{T}, \trianglelefteq, f_r) \quad \text{iff} \quad r > c.$$

Through investigations on ordinal notation systems, e.g. with regard to phase transition, Weiermann has tried to give a possible approach to the question like what the intrinsic properties that distinguish some orderings from the others are or if the systems are independent of their possible use in proof-theoretic work. Cf. [14,16,17] for his pioneer works on phase transition in logic. In this sense it is a natural consequence that the structurally equivalent systems show the same behavior in view of slowly-well-orderedness, see Section 4.

Notational conventions Small Latin letters $i, m, n, \ldots$ range over natural numbers while Greek letters $\alpha, \beta, \ldots$ range over ordinals or finite trees. Given a non-negative real number x, $\lfloor x \rfloor$ is the largest natural number not bigger than x and $\lceil x \rceil$ is the least natural number not less than x. Define

$$|x| := \lceil \log_2(x+1) \rceil,$$

i.e., the length of the binary representation of x. We iterate the $|\cdot|$-function: $|x|_0 := x$ and $|x|_{m+1} := ||x|_m|$. And inv is the inverse function of the super-exponential function:

$$\mathrm{inv}(i) = \min\{m \colon |i|_m \leq 1\}$$

A function $f \colon \mathbb{N} \to \mathbb{N}$ is said to be *unbounded* if it is weakly monotone-increasing and the values are not bounded. For any unbounded function f its *inverse* function is defined as follows:

$$f^{-1}(i) := \min\{\ell \colon i < f(\ell)\}$$

Note that $f^{-1}(i) \leq \ell$ if and only if $i < f(\ell)$.

2 An difference

We start with two combinatorial results of Weiermann. They indicate that the system based on binary trees is of different character than Cantor's system. His works are based on Friedman-style miniaturization. He found out that they behave differently in view of the slowly-well-orderedness.

2.1 Cantor system

For any nonzero ordinal $\alpha < \varepsilon_0$ there exist an unique natural number n and uniquely determined ordinals $\alpha_1, \ldots, \alpha_n < \varepsilon_0$ such that $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ and $\alpha > \alpha_1 \geq \cdots \geq \alpha_n$. It is denoted by $\alpha =_{NF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ and said to be in *Cantor normal form.*

Given α, $\beta \in \varepsilon_0$ put

$$\alpha_0(\beta) := \beta, \quad \alpha_{n+1}(\beta) := \alpha^{\alpha_n(\beta)}, \quad \text{and} \quad \alpha_n := \alpha_n(1).$$

The fundamental sequence for a limit λ is defined as above. For convenience, we set also $(\alpha + 1)[n] := \alpha$ for any n and $\varepsilon_0[n] = \omega_{n+1}$. The *norm* $N\alpha$ is the number of occurrences of ω in α: $N0 := 0$ and if $\alpha =_{NF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$, then

$$N\alpha := n + N\alpha_1 + \cdots + N\alpha_n.$$

Definition 5 (Slowly-well-orderedness, SWO) *Let $(X, \preceq)$ be a linear ordering.*

(1) A function $g : X \to \mathbb{N}$ is called a norm function *if for every $n \in \mathbb{N}$ the set $\{\alpha \in X : g(\alpha) \leq n\}$ is finite.*
(2) Given a norm function $g : X \to \mathbb{N}$ and a function $f : \mathbb{N} \to \mathbb{N}$ the slowly-well-orderedness *of $(X, \preceq)$ is given by* $\mathrm{SWO}(X, \preceq, f, g) :\equiv$

> *For any k there exists a constant n which is so large that, for any finite sequence of ordinals $\alpha_0, \ldots, \alpha_n \in X$ with $g(\alpha_i) \leq k + f(i)$ for all $i \leq n$, there exist indices ℓ and m such that $\ell < m \leq n$ and $\alpha_\ell \preceq \alpha_m$.*

Set $\mathrm{SWO}(\varepsilon_0, <, f) := \mathrm{SWO}(\varepsilon_0, <, f, N)$. Note that $\mathrm{SWO}(\varepsilon_0, <, f)$ is a Π^0_2 if f is e.g. a primitive recursive function.

Theorem 6 (Friedman [11,12]) $\mathrm{SWO}(\varepsilon_0, <, id)$ *is not* PA*-provable.*

Using this, Weiermann and Arai gave an interesting characterization of the class of functions f such that $\mathrm{SWO}(\varepsilon_0, <, f)$ is not PA-provable.

Theorem 7 (Weiermann [14]) *Let $m \in \mathbb{N}$.*

(1) $\mathrm{SWO}(\varepsilon_0, <, f_0)$ *is* PRA*-provable for $f_0(i) := |i| \cdot \mathrm{inv}(i)$.*
(2) $\mathrm{SWO}(\varepsilon_0, <, f_1)$ *is not* PA*-provable for $f_1(i) := |i| \cdot |i|_m$.*

Let $L(\cdot; F_\alpha^{-1})$ be defined by

> the least n such that for any finite sequence of ordinals $\alpha_0, \ldots, \alpha_n < \varepsilon_0$ with $N\alpha_i \leq k + |i| \cdot |i|_{F_\alpha^{-1}(i)}$ for all $i \leq n$, there exist indices ℓ and m such that $\ell < m \leq n$ and $\alpha_\ell \leq \alpha_m$.

Theorem 8 (Arai [18]) *Let $\alpha \leq \varepsilon_0$.*

(1) $L(\cdot; F_\alpha^{-1})$ is primitive recursive in F_α and vice versa. Therefore, $L(\cdot; F_\alpha^{-1})$ is provably total in PA *iff $\alpha < \varepsilon_0$.*
(2) Let $f_\alpha(i) = |i| \cdot |i|_{F_\alpha^{-1}(i)}$. Then

$$\mathrm{SWO}(\varepsilon_0, <, f_\alpha) \textit{ is PA-provable iff } \alpha < \varepsilon_0.$$

The system $(\mathcal{B}, <)$ obtained from the Feferman-Schütte notation system[2] for Γ_0 by omitting the operator '+' constitutes a well-ordering of order type ε_0. Moreover, $\leq$ is a canonical extension of the homeomorphic embeddability relation $\trianglelefteq$ on the set of rooted binary trees: A *rooted binary tree* T is a set of nodes such that, if it is not empty, there is one distinguished node called the root of T and the remaining nodes are partitioned into two rooted binary trees. The following definition of the set $\mathcal{B}$ of all rooted binary trees is very useful for our study.

Assume that a constant symbol o and a binary function symbol φ are given. In the following, $\varphi\alpha\beta$ stands for $\varphi(\alpha, \beta)$.

Definition 9 *The set $\mathcal{B}$ and the homeomorphic embeddability relation $\trianglelefteq$ are defined inductively as follows:*

(1) $o \in \mathcal{B}$ and $o \trianglelefteq \beta$ for all $\beta \in \mathcal{B}$;
(2) if α_1, $\beta_1 \in \mathcal{B}$, then $\varphi(\alpha_1, \beta_1) \in \mathcal{B}$, and $\varphi\alpha_1\beta_1 \trianglelefteq \varphi\alpha_2\beta_2$ if
  (a) $\varphi\alpha_1\beta_1 \trianglelefteq \alpha_2$ or $\varphi\alpha_1\beta_1 \trianglelefteq \beta_2$; or
  (b) $\alpha_1 \trianglelefteq \beta_1$ and $\alpha_2 \trianglelefteq \beta_2$.

Theorem 10 (Higman [23]) $(\mathcal{B}, \trianglelefteq)$ *is a well-partial ordering.*

Given a well-partial ordering $(X, \preceq)$ define its *maximal order type* by

$$o(X, \preceq) := \sup\{otype(\prec^+) \colon \prec^+ \text{ is a well-ordering on } X \text{ extending } \prec\},$$

where $otype(\prec^+)$ denotes the order type of the well-ordering $\prec^+$.

Theorem 11 (de Jongh and Parikh [24]) *Assume $(X, \preceq)$ is a well-partial ordering. Then there is a well-ordering $\prec^+$ extending $\preceq$ such that $o(X, \preceq) = otype(\prec^+)$.*

In case of $(\mathcal{B}, \trianglelefteq)$ we easily find such a well-ordering $<$ on $\mathcal{B}$: $<$ is the least binary relation on $\mathcal{B}$ defined as follows:

(1) if $\alpha = o$ and $\beta \neq o$, then $\alpha < \beta$;
(2) if $\alpha = \varphi\alpha_1\alpha_2$ and $\beta = \varphi\beta_1\beta_2$, then $\alpha < \beta$ if one of the following hold:
  (a) $\alpha_1 < \beta_1$ and $\alpha_2 < \beta$;
  (b) $\alpha_1 = \beta_1$ and $\alpha_2 < \beta_2$; or
  (c) $\alpha_1 > \beta_1$ and $\alpha \leq \beta_2$.

The following is a folklore in proof theory.

[2] See e.g. [19–22]

Lemma 12 *$<$ is a well-ordering on $\mathcal{B}$ extending $\trianglelefteq$, and otype$(<) = \varepsilon_0$.*

The following unpublished result by de Jongh is well known. For further details see, for example, Schmidt [25].

Theorem 13 (de Jongh) $o(\mathcal{B}, \trianglelefteq) = \varepsilon_0$.

We want to here present an alternative, relatively simple proof of the theorem. A cumulative hierarchy is used.

Given a natural number d we define $\mathcal{B}^d$ recursively as follows:

- $o \in \mathcal{B}^d$;
- if $d > 0$, $\alpha \in \mathcal{B}^{d-1}$, and $\beta \in \mathcal{B}^d$, then $\varphi\alpha\beta \in \mathcal{B}^d$.

And define $\rho^d(\alpha)$ for $\alpha \in \mathcal{B}$ as follows:

- $\rho^0(\alpha) = \alpha$;
- $\rho^{d+1}(\alpha) = \varphi\rho^d(\alpha)0$.

Lemma 14 *Let $d \in \mathbb{N}$.*

(1) $\mathcal{B} = \bigcup\{\mathcal{B}^d : d \in \omega\}$.
(2) If $\alpha \in \mathcal{B}^d$, then $\alpha < \rho^{d+1}(o)$ and $\rho^k(\alpha) \in \mathcal{B}^{d+k}$.
(3) $\rho^{d+1}(o) \in \mathcal{B}^{d+1} \setminus \mathcal{B}^d$.
(4) If $\alpha < \beta$, then $\rho^d(\alpha) < \rho^d(\beta)$.
(5) If $\alpha \trianglelefteq \beta$, then $\rho^d(\alpha) \trianglelefteq \rho^d(\beta)$.
(6) If $\alpha \in \mathcal{B}^{d+1} \setminus \mathcal{B}^d$ and $\beta \in \mathcal{B}^d$, then $\alpha \not\trianglelefteq \beta$ and $\beta < \alpha$.

PROOF. The first five claims are obvious. The last one will be shown by induction on β. If $\beta = 0$ there is nothing to show. Let $\alpha = \varphi\alpha_1\alpha_2$ and $\beta = \varphi\beta_1\beta_2$. If $\alpha_1 \in \mathcal{B}^{d-1}$, then $\alpha_2 \in \mathcal{B}^{d+1} \setminus \mathcal{B}^d$. Hence $\beta < \alpha_2 < \alpha$ by I.H. Now assume $\alpha_1 \in \mathcal{B}^d \setminus \mathcal{B}^{d-1}$. Then $\beta_1 < \alpha_1$ and $\beta_2 < \alpha$ by I.H., so $\beta < \alpha$ and $\alpha \not\trianglelefteq \beta$. $\square$

We are now going to compute $o(\mathcal{B}^d)$ by comparing $(\mathcal{B}^d, \trianglelefteq)$ with a well-known well-partial ordering. Higman's Lemma plays an important role.

Definition 15 *Let $(A, \preceq)$ be a partial ordering and A^* be the set of finite lists of members of A. The* Higman embedding *$\preceq_h$ is a partial ordering on A^* defined by:*

$$a_1 \cdots a_m \preceq_h b_1 \cdots b_n$$

if there is a strictly increasing function $g : [1, m] \to [1, n]$ such that $a_i \preceq b_{g(i)}$ for any $i \in \{1, \ldots, m\}$.

Theorem 16 (Higman's Lemma)

(1) If A is a well-partial ordering, then A^* is a well-partial ordering with respect to the Higman embedding.
(2) If $(A, \preceq)$ is a well-partial ordering with $o(A, \preceq) = \alpha$, then

$$o(A^*, \preceq_h) = \begin{cases} \omega^{\omega^{\alpha-1}} & \text{if } \alpha \in \omega \setminus \{0\}, \\ \omega^{\omega^{\alpha}} & \text{if } \alpha = \beta + m, \text{ where } \beta \geq \omega,\ \beta \neq \omega^{\beta}, \text{ and } m \in \omega, \\ \omega^{\omega^{\alpha+1}} & \text{otherwise.} \end{cases}$$

PROOF. See e.g. [24–26]. □

Note that there is a similarity between $\mathcal{B}^{d+1}$ and $(\mathcal{B}^d)^*$. In fact, every $\alpha \in \mathcal{B}^{d+1}$ is of the form $\alpha = \varphi\alpha_1\varphi\alpha_2 \cdots \varphi\alpha_m o$, where $\alpha_i \in \mathcal{B}^d$. If $\beta = \varphi\beta_1\varphi\beta_2 \cdots \varphi\beta_n o \in \mathcal{B}^{d+1}$ and $\alpha_1 \cdots \alpha_m \trianglelefteq_h \beta_1 \cdots \beta_n$ then $\alpha \trianglelefteq \beta$. Unfortunately the relation above is not isomorphic. Nevertheless, we will show that $o(\mathcal{B}^{d+1}, \trianglelefteq) = o((\mathcal{B}^d)^*, \trianglelefteq_h)$ with the aid of the following obvious lemma.

Lemma 17 *Let $(A, \preceq_1)$ and $(B, \preceq_2)$ be well-partial orderings and $f : A \to B$ an injective function such that*

$$a \preceq_1 b \quad \textit{iff} \quad f(a) \preceq_2 f(b)$$

for all a, $b \in A$. Then $o(A, \preceq_1) \leq o(B, \preceq_2)$.

Theorem 18 *For any $d > 0$, $o(\mathcal{B}^{d+1}, \trianglelefteq) = o(\mathcal{B}^{d+1} \setminus \mathcal{B}^d, \trianglelefteq) = o((\mathcal{B}^d)^*, \trianglelefteq_h)$.*

PROOF. Consider $f : \mathcal{B}^{d+1} \to (\mathcal{B}^d)^*$ and $g : (\mathcal{B}^d)^* \to \mathcal{B}^{d+1} \setminus \mathcal{B}^d$ defined by:

$$f(\alpha) := \begin{cases} \emptyset & \text{if } \alpha = o, \\ \alpha & \text{if } \alpha = \varphi\alpha_1\alpha_2 \in \mathcal{B}^d, \\ \alpha, \alpha_1, f(\alpha_2) & \text{if } \alpha = \varphi\alpha_1\alpha_2 \notin \mathcal{B}^d \end{cases}$$

and

$$g(\alpha_1, \ldots, \alpha_m) := \varphi\alpha_1\varphi\alpha_2 \cdots \varphi\alpha_m \rho^{d+1}(o),$$

where $\emptyset$ denotes the empty sequence. It suffices to show that f and g satisfy the conditions in Lemma 17.

(1) Let $\alpha, \beta \in \mathcal{B}^{d+1}$.

(a) Assume $\alpha \trianglelefteq \beta$. We show $f(\alpha) \trianglelefteq_h f(\beta)$ by induction on the number of occurrences of φ in α and β. If $\alpha \in \mathcal{B}^d$, then it is obvious. Assume now

$\alpha = \varphi\alpha_1\alpha_2 \notin \mathcal{B}^d$. Then $\beta = \varphi\beta_1\beta_2 \notin \mathcal{B}^d$ for some β_1, β_2 by Lemma 14. Furthermore, we have $\alpha \not\trianglelefteq \beta_1$ since $\beta_1 \in \mathcal{B}^d$. Hence there are two possible cases, i.e. either $\alpha \trianglelefteq \beta_2$ or $\alpha_i \trianglelefteq \beta_i$, $i = 1, 2$. In any case the claim follows from the I.H. since $f(\alpha) = \alpha, \alpha_1, f(\alpha_2)$ and $f(\beta) = \beta, \beta_1, f(\beta_2)$.

(b) Assume $f(\alpha) \trianglelefteq_h f(\beta)$. We show $\alpha \trianglelefteq \beta$ by induction on the number of occurrences of φ in α and β. If $\alpha \in \mathcal{B}^d$, then it is again obvious since each component in $f(\beta)$ is a subtree of β. Assume $\alpha \notin \mathcal{B}^d$, then $\beta \notin \mathcal{B}^d$ and $\alpha \not\trianglelefteq \beta_1$. Since then $f(\beta) = \beta, \beta_1, f(\beta_2)$ we have either $\alpha \trianglelefteq \beta$ or $f(\alpha) \trianglelefteq_h f(\beta_2)$. In the latter case the I.H. implies $\alpha \trianglelefteq \beta_2 \trianglelefteq \beta$.

(2) Let $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \in \mathcal{B}^d$.

(a) Assume $\alpha_1, \ldots, \alpha_m \trianglelefteq_h \beta_1, \ldots, \beta_n$. Then there is an increasing one-to-one function $\mu : m \to n$ such that $\alpha_i \trianglelefteq \beta_{\mu(i)}$ for any $i = 1, \ldots, m$. Then we can easily show that $g(\alpha_1, \ldots, \alpha_m) \trianglelefteq g(\beta_1, \ldots, \beta_n)$.

(b) Assume $\varphi\alpha_1 \cdots \varphi\alpha_m\rho^{d+1}(o) \trianglelefteq \varphi\beta_1 \cdots \varphi\beta_n\rho^{d+1}(o)$. By induction on n we show that $\alpha_1, \ldots, \alpha_m \trianglelefteq_h \beta_1, \ldots, \beta_n$. Assume $n = 0$. Then $m = 0$ since $\varphi\gamma\rho^{d+1}(o) \not\trianglelefteq \rho^{d+1}(o)$ for any $\gamma \in \mathcal{B}$. Assume now $n > 0$. If $m = 0$, it is obvious. If $m > 0$, then there must be some $j \in \{1, \ldots, n\}$ such that $\alpha_1 \trianglelefteq \beta_j$ and $\varphi\alpha_2 \cdots \varphi\alpha_m\rho^{d+1}(o) \trianglelefteq \varphi\beta_{j+1} \cdots \varphi\beta_n\rho^{d+1}(o)$. This is because $\rho^{d+1}(o) \notin \mathcal{B}^d$. The claim follows now by I.H. □

Corollary 19 *For any $d > 0$, $(\mathcal{B}^d, \trianglelefteq)$ is a well-partial ordering and $o(\mathcal{B}^d, \trianglelefteq) = \omega_{2d-1}(1)$.*

PROOF. If $d = 1$, then $\mathcal{B}^1 = \{o, \varphi oo, \varphi o(\varphi oo), \ldots\}$ is linearly ordered by $\trianglelefteq$, so $o(\mathcal{B}^1, \trianglelefteq) = \omega$. If $d > 1$, use I.H., Theorem 18, and Higman's Lemma. □

Theorem 20 $o(\mathcal{B}, \trianglelefteq) = \varepsilon_0$.

PROOF. By Corollary 19 we have $o(\mathcal{B}, \trianglelefteq) \geq \varepsilon_0$. Assume $o(\mathcal{B}, \trianglelefteq) > \varepsilon_0$. Then there would be a well-ordering $\prec$ on $\mathcal{B}$ extending $\triangleleft$ such that $otype(\triangleleft) > \varepsilon_0$. This is, however, impossible since each restriction of $\prec$ to $\mathcal{B}^d$ has the order type less than ε_0 again by Corollary 19. □

Let the *norm* of $\alpha \in \mathcal{B}$, $\|\alpha\|$, be the number of occurrences of φ in α, i.e.

$$\|o\| = 0 \quad \text{and} \quad \|\varphi\alpha\beta\| = 1 + \|\alpha\| + \|\beta\|$$

It is evident that $\mathcal{B}$, $\trianglelefteq$, $<$, and $\|\cdot\|$ are all primitive recursively definable in PA.

Definition 21 (Slowly-well-partial-orderedness, SWP) *Let* $(X, \preceq)$ *be a partial ordering. Then given a norm function* $g : X \to \mathbb{N}$ *and a function* $f : \mathbb{N} \to \mathbb{N}$ *the* slowly-well-partial-orderedness *is given by* $\mathrm{SWP}(X, \preceq, f, g) :\equiv$

> *For any* k *there exists a constant* n *which is so large that, for any finite sequence* $\alpha_0, \ldots, \alpha_n$ *of finite trees with* $|\alpha_i| \leq k + f(i)$ *for all* $i \leq n$*, there exist indices* $\ell < m \leq n$ *satisfying* $\alpha_\ell \preceq \alpha_m$*.*

$\mathrm{SWP}(\mathcal{B}, \preceq, f)$ *stands for* $\mathrm{SWP}(\mathcal{B}, \preceq, f, \|\cdot\|)$ *in case of* $\preceq \in \{\trianglelefteq, <\}$*.*

Theorem 22 (Weiermann [27]) *Given a primitively recursive real number* r *set* $f_r(i) := r \cdot |i|$*. And let* $\preceq \in \{\trianglelefteq, <\}$*. Then it holds that*

$$\mathrm{SWP}(\mathcal{B}, \preceq, f_r) \textit{ is } \mathrm{PA}\textit{-provable if } r \leq \frac{1}{2}\,.$$

The proof is not published yet. But Section 4 in [14] indicates the existence of such a constant like $\frac{1}{2}$. The proof in [27] is very similar. It uses just another counting method, hence a different constant concerning the binary trees than Otter's tree constant which is specific for general finite trees.

Remark 23 *The theorem above indicates that Cantor's system for* ε_0 *is of different character than the system* $(\mathcal{B}, <)$*. In fact, we know by Theorem 7 that* $\mathrm{SWO}(\varepsilon_0, <, \lambda x\,.\,r|x|)$ *is* PA*-provable for any* r*, while* $\mathrm{SWO}(\mathcal{B}, <, \lambda x\,.\,r|x|)$ *is not* PA*-provable if* $r > \frac{1}{2}$*.*

3 Structural equivalences

We now show that the systems except $(\mathcal{B}, <)$ are structurally equivalent. The functions themselves which establish the structural equivalences look very simple and somehow canonical while the proofs of the equivalences are not so quite obvious. We begin with one of the most recent systems.

3.1 Graded provability algebra and Beklemishev's system

Let T be an elementarily represented, sound fragment of PA containing $\mathrm{I}\Sigma_1$. The *Lindenbaum boolean algebra* $\mathcal{L}_\mathrm{T}$ is the set of all sentences modulo provable equivalence in T.

Let n-$\mathsf{Con}(\mathrm{T})$ denote a natural formula expressing that the theory $\mathrm{T}+Th_{\Pi_n}(\mathbb{N})$ is consistent, where $Th_{\Pi_n}(\mathbb{N})$ is the set of all true arithmetical Π_n sentences. The *graded provability algebra* of T, $\mathcal{M}_\mathrm{T}$, is the structure of Lindenbaum boolean algebra $\mathcal{L}_{\mathrm{T}}$ with the *n-consistency operator* $\langle n\rangle_{\mathrm{T}}$, $n \in \mathbb{N}$, defined by $\langle n\rangle_{\mathrm{T}}\varphi := n\text{-}\mathsf{Con}(\mathrm{T}+\varphi)$.

The subscript T will be suppressed if the underlying theory is known from the context. The *n-provability operator* $[n]$ is defined by $[n]\varphi := \neg\langle n\rangle\neg\varphi$. $\langle 0\rangle\varphi$ is usually written by $\Diamond\varphi$ and $[0]\varphi$ by $\Box\varphi$. Terms of the graded provability algebra correspond to propositional polymodal formulas.

GLP based on the identities of $\mathcal{M}_{\mathrm{T}}$ is an extension of the Gödel-Löb system GL:[3] Let m, $n \in \mathbb{N}$.

(1) **Axioms**
- Boolean tautologies
- $\langle n\rangle(\varphi \vee \psi) \to (\langle n\rangle\varphi \vee \langle n\rangle\psi)$
- $\neg\langle n\rangle\neg\top$
- $\langle n\rangle\varphi \to \langle n\rangle(\varphi \wedge \neg\langle n\rangle\varphi)$
- $\langle n\rangle\varphi \to \langle m\rangle\varphi$ for $m \le n$
- $\langle m\rangle\varphi \to [n]\langle m\rangle\varphi$ for $m < n$

(2) **Rules**
- modus ponens
- $\varphi \to \psi \vdash \langle n\rangle\varphi \to \langle n\rangle\psi$

Then it holds that $\mathrm{GLP} \vdash \varphi(\vec{x})$ iff $\mathcal{M}_{\mathrm{T}} \models \forall\vec{x}(\varphi(\vec{x}) = \top)$.

Let S be the set of all finite words in the alphabet $\mathbb{N}$, including the empty word Λ. S_n is the restriction of S to the alphabet $\{n, n+1, \dots\}$. We identify each element $\alpha = n_1 \cdots n_k$ of S with its *modal interpretation* $\langle n_1\rangle \cdots \langle n_k\rangle\top$.

We write $\alpha \sim \beta$ if $\mathrm{GLP} \vdash \alpha \leftrightarrow \beta$. And $\alpha = \beta$ means the graphical identity. The orderings $<_n$ are defined on S by:

$$\alpha <_n \beta \quad \text{iff} \quad \mathrm{GLP} \vdash \beta \to \langle n\rangle\alpha.$$

Note that $<_n$ are transitive and irreflexive because of Gödel's incompleteness theorem. Below we summarize some results from Beklemishev [5].

Given $\alpha \in S$ let α^k denote the k times iterated concatenation of α. The function $o\colon S \to \varepsilon_0$ is given as follows:

- $o(0^k) = k$;
- if $\alpha = \alpha_0 0 \cdots 0\alpha_n$, where all $\alpha_i \in S_1$ and not all of them empty, then

$$o(\alpha) = \omega^{o(\alpha_n^-)} + \cdots + \omega^{o(\alpha_0^-)}.$$

Here γ^- is obtained from $\gamma \in S_1$ by replacing every letter $m+1$ with m.

[3] Cf. Boolos [28] for more about provability logic

Note that some of the elements of S are pairwise equivalent. However, there is a set of elements which represent each equivalence class, namely the set NF of words in *normal form*. We define $\alpha \in NF$ by recursive induction on the *width* $w(\alpha)$, i.e. the number of different letters occurring in α.

- if $w(\alpha) \leq 1$, then $\alpha \in NF$;
- assume $w(\alpha) > 2$ and let n be the smallest letter in α such that graphically $\alpha = \alpha_0 n \cdots n \alpha_k$, where all $\alpha_i \in S_{n+1}$. Then $\alpha \in NF$ if all $\alpha_i \in NF$ and $\alpha_{i+1} \not<_{n+1} \alpha_i$ for any $i < k$.

Theorem 24 (Beklemishev) *Let* α, $\beta \in S$.

(1) $(S, <_0)$ *is a well-partial ordering of height* ε_0.
(2) Every word $\alpha \in S$ *has an uniquely defined equivalent normal form.*
(3) If $\alpha \sim \beta$ *then* $o(\alpha) = o(\beta)$.
(4) If $\alpha <_0 \beta$ *then* $o(\alpha) < o(\beta)$.
(5) $o \restriction NF \colon NF \to \varepsilon_0$ *is an order-preserving isomorphism.*

It is also possible to assign fundamental sequences to each element of S. For $\alpha \in S$ and $k \in \mathbb{N}$ we define $\alpha[k] \in S$ as follows:

- if $\alpha = \langle 0 \rangle \beta$ then $\alpha[k] = \beta$;
- if $\alpha = \langle n+1 \rangle \gamma \langle m \rangle \beta$, where $\gamma \in S_{n+1}$ and $m \leq n$, then $\alpha[k] = (n\gamma)^{k+1} m \beta$.

There will be no confusion with the notation for fundamental sequences with respect to the ordinals up to ε_0.

Theorem 25 (Beklemishev) *Let* $\alpha = \langle n+1 \rangle \beta$ *and* $k \in \mathbb{N}$.

(1) If $\alpha \in NF$, *then* $\alpha[k] \in NF$.
(2) $\alpha[k] <_0 \alpha[k+1] <_0 \alpha$.
(3) For every $\beta \in S$ *with* $\beta <_0 \alpha$ *there is an* $\ell \in \mathbb{N}$ *such that* $\beta <_0 \alpha[\ell]$.

3.2 Worms, Hydras, and tree-ordinals

The *Hydra battle* introduced by Kirby and Paris in [29] has an isomorphic formulation in terms of ordinals, namely fundamental sequences for ordinals below ε_0. Let $\cdot[\cdot]$ denote the standard assignments of fundamental sequences and let $\alpha(0) = \alpha$ and $\alpha(i+1) = \alpha(i)[i]$. Then the fact that chopping off the rightmost head is a winning strategy for Hercules is formalized by:

$$\text{for any } n \text{ there exists an } i \text{ such that } \omega_n(i) = 0$$

This is a true Π^0_2-sentence because of the well-foundedness, but PA-unprovable since $H_\alpha(0) \leq \min\{i \colon \alpha(i) = 0\}$.[4]

Another similar combinatorial game is introduced by Beklemishev in [30] as an application of proof-theoretic analysis to his ordinal notation system S. It deals with objects called *worms* and is hence called *Worm principle*. See also Carlucci [31] where the isomorphism of the Worm and the Hydra battle was established (independently of this paper).

A *worm* is just a finite function with natural numbers as values. We identify the worm $f \colon [0, n] \to \mathbb{N}$ with the list $f(0) \cdots f(n)$ or $\langle f(0), \ldots, f(n) \rangle$. We call $f(n)$ the *head* of the worm. $\emptyset$ denotes the empty function. Let W be the set of all worms and W_n the subset of W whose elements have values $\geq n$.

A Worm game begins with a worm and at each step we chop off its head. In response the worm grows in length according to some rules. Formally, we specify a function $next \colon W \times \mathbb{N} \to W$. Let α range over worms.

(1) $next(\emptyset, k) := \emptyset$.
(2) Let $\alpha = a_0 \cdots a_n$.
 - If $a_n = 0$, then $next(\alpha, k) := a_0 \cdots a_{n-1}$.
 - If $a_n > 0$, let $m := \max\{i < n \colon a_i < a_n\}$. We define

$$next(\alpha, k) := r * \underbrace{s * s * \cdots * s}_{k+1 \text{ times}} .$$

 where $r = \langle a_0, \ldots, a_m \rangle$, $s = \langle a_{m+1}, \ldots, a_{n-1}, a_n - 1 \rangle$.

Here $*$ means the concatenation function of worms. Now let $\alpha(0) := \alpha$ and $\alpha(n+1) := next(\alpha(n), n+1)$. Then the *Worm principle* says that **E**very **W**orm **D**ies:

$$\textbf{EWD} := \text{ for any worm } \alpha \text{ there exists an } n \text{ such that } \alpha(n) = \emptyset$$

Note that **EWD** is a Π^0_2-sentence since $\alpha(n)$ is defined primitive recursively and that the size of maximal element of worms cannot increase. Hence $\alpha(n) = \beta$ can be written out as a Δ_0 formula in three variables.

Theorem 26 (Beklemishev [30])

(1) **EWD** *is true, but not* PA*-provable.*
(2) **EWD** *is* PA*-equivalent to* 1-Con(PA).

What is responsible for the PA-unprovability of **EWD**? A proof-theoretical explanation is that the Skolem function of **EWD** should grow too fast to be

[4] Cf. Fairtlough and Wainer [7].

provably total in PA. Below we give a characterization of the growth rate conditions responsible for the too fast growth. In order to emphasize the relevance to S we prefer another notation to $next(\alpha, k)$.

Definition 27 *Let α, β, $\gamma \in W$.*

$$\alpha[\![k]\!] := \begin{cases} \emptyset & \text{if } \alpha = \emptyset, \\ \beta & \text{if } \alpha = \beta 0, \\ \beta m (\gamma n)^{k+1} & \text{if } \alpha = \beta m \gamma \langle n+1 \rangle, \ \gamma \in W_{n+1}, \ m \leq n. \end{cases}$$

Note that $next(\alpha, k) = \alpha[\![k]\!]$. Given $f\colon \mathbb{N} \to \mathbb{N}$ and $\alpha \in W$ set

$$\alpha(f, 0) := \alpha, \quad \alpha(f, n+1) := \alpha(f, n)[\![f(n+1)]\!],$$

and define

$$\mathbf{EWD}(f) :\equiv \forall \alpha\, \exists n\, (\alpha(f, n) = \emptyset).$$

Then $\mathbf{EWD} = \mathbf{EWD}(id)$. And $\mathbf{EWD}(f)$ remains true Π^0_2-sentence if f is primitive recursive.

To analyze the growth rate of the Skolem function of $\mathbf{EWD}(f)$ in terms of fast growing hierarchies one should notice that the function o between S and the Cantor system for ε_0 defined above cannot be used in its original form since the correspondence is not one-to-one. This is why we turn our attention to the tree-ordinals.

Definition 28 *The set Ω of countable* tree-ordinals *is generated inductively as follows:*

- $0 \in \Omega$*;*
- *if* $\alpha \in \Omega$*, then* $\alpha + 1 := \alpha \cup \{\alpha\} \in \Omega$*;*
- *if* $\alpha_n \in \Omega$ *for all* $n \in \mathbb{N}$*, then* $\alpha := \langle \alpha_n \rangle_{n \in \mathbb{N}} \in \Omega$*.*

λ will always denote a *limit* of the form $\lambda = \langle \lambda_n \rangle_n := \langle \lambda_n \rangle_{n \in \mathbb{N}}$. Addition, multiplication, and exponentiation are defined as usual:

- **Addition:** $\alpha + 0 := \alpha$; $\alpha + (\beta + 1) := (\alpha + \beta) + 1$; $\alpha + \lambda := \langle \alpha + \lambda_n \rangle_n$
- **Multiplication:** $\alpha \cdot 0 := 0$; $\alpha \cdot (\beta + 1) := (\alpha \cdot \beta) + \alpha$; $\alpha \cdot \lambda := \langle \alpha \cdot \lambda_n \rangle_n$
- **Exponentiation:** $\alpha^0 := 1$; $\alpha^{(\beta+1)} := \alpha^\beta \cdot \alpha$; $\alpha^\lambda := \langle \alpha^{\lambda_n} \rangle_n$

There is a set $\mathbb{T} \subseteq \Omega$ whose elements correspond to ordinals up to ε_0. Set

$$n := 0 + \underbrace{1 + \cdots + 1}_{n \text{ times}} \quad \text{and} \quad \omega := \langle 1 + n \rangle_n \,.$$

Maybe it is an abuse of symbols to use the same names such as n and ω. However, it will always be clear from the context to which world they belong.

Definition 29 $\mathbb{T}$ *is defined inductively as follows:*

- $0 \in \mathbb{T}$*;*
- *if* $\alpha_0, \ldots, \alpha_n \in \mathbb{T}$*, then also* $\omega^{\alpha_0} + \cdots + \omega^{\alpha_n} \in \mathbb{T}$*.*

Note that each tree-ordinal in $\mathbb{T}$ represents a unique determined (ordered) tree figure other than the ordinals in the Cantor system. And though there is a canonical fundamental sequence for each limit tree-ordinal, we shall make some modifications for technical reasons. This modifications will have no significant effect on the fast growing hierarchy we consider. We write $\alpha{\cdot}m$ for $\underbrace{\alpha + \cdots + \alpha}_{m \text{ times}}$.

Definition 30 (Fundamental sequences for tree-ordinals) *Let* $\alpha \in \mathbb{T}$*.*

- *If* $\alpha = 0$*, then* $\alpha[k] = 0$*.*
- *If* $\alpha = \beta + 1$*, then* $\alpha[k] = \beta$*.*
- *If* $\alpha = n + \omega$ *for some* $n \in \mathbb{N}$*, then* $\alpha[k] = n + k + 1$*.*
- *If* $\alpha = \beta + \omega$ *and* $\beta \neq n$ *for any* $n \in \mathbb{N}$*, then* $\alpha[k] = \beta + k + 2$*.*
- *If* $\alpha = \beta + \omega^{\gamma+1}$ *and* $\gamma \neq 0$*, then* $\alpha[k] = \beta + \omega^{\gamma} \cdot (k+1) + 1$*.*
- *If* $\alpha = \beta + \omega^{\lambda}$ *and* λ *a limit, then* $\alpha[k] = \beta + \omega^{\lambda[k]}$*.*

Definition 31 *The* sub-tree ordering $\prec$ *is the transitive closure of the rule:*

$$\alpha[m] \prec \alpha \text{ for all } \alpha \in \mathbb{T} \setminus \{0\} \text{ and } m \in \mathbb{N}.$$

That the sub-tree ordering build a well-ordering can be proven easily as in Fairtlough and Wainer [7].

Theorem 32 *The set* $\{\beta \mid \beta \prec \alpha\}$ *is well-ordered by* $\prec$ *and of order type less than* ε_0*.*

Once more we shall use the notation o for an isomorphism between worms and tree-ordinals from $\mathbb{T}$.

Definition 33 $o \colon W \to \mathbb{T}$ *is defined recursively as follows:*

- $o(0^k) := k$*;*
- *if* $\alpha = \alpha_0 0 \cdots 0 \alpha_n$*, where all* $\alpha_i \in W_1$ *and not all of them empty, then*

$$o(\alpha_0 0 \alpha_1 0 \cdots 0 \alpha_n) := \omega^{o(\alpha_0^-)} + \cdots + \omega^{o(\alpha_n^-)}.$$

There will be no confusion between $o : W \to \mathbb{T}$ and $o : S \to \varepsilon_0$. One of the main differences is, however, that $o \colon W \to \mathbb{T}$ is one-to-one and onto. The function $g \colon \mathbb{T} \to W$ defined by

- $g(k) := 0^k$;

- if $\alpha = \omega^{\alpha_0} + \cdots + \omega^{\alpha_n}$ and $\alpha_i \neq 0$ for some $i \leq n$, then

$$g(\alpha) = g(\alpha_0)^+ 0 \cdots 0 g(\alpha_n)^+,$$

is obviously the inverse function of $o : W \to \mathbb{T}$, where β^+ is obtained from $\beta \in W$ by replacing every letter m with $m + 1$.

Lemma 34 *Let α, $\beta \in W$. Then*

$$o(\alpha 0 \beta) = \begin{cases} o(\alpha) + 1 + o(\beta) & \text{if } 0^m \in \{o(\alpha), o(\beta)\} \text{ for some } m \in \mathbb{N}, \\ o(\alpha) + o(\beta) & \text{otherwise.} \end{cases}$$

PROOF.

(1) Let $\alpha = 0^m$ and $\beta = 0^n$ for some m, $n \in \omega$. Then

$$o(\alpha 0 \beta) = o(0^{m+1+n}) = m + 1 + n = o(\alpha) + 1 + o(\beta).$$

(2) Let $\alpha = 0^m$ for some $m \in \omega$ and $\beta = \beta_0 0 \cdots 0 \beta_n$, where all $\beta_j \in W_1$ and not all of them empty. Then

$$\begin{aligned} o(\alpha 0 \beta) &= o(0^{m+1} \beta_0 0 \cdots 0 \beta_n) \\ &= m + 1 + \omega^{o(\beta_0^-)} + \cdots + \omega^{o(\beta_n^-)} \;=\; o(\alpha) + 1 + o(\beta). \end{aligned}$$

(3) Similar for the case that $\beta = 0^n$ for some $n \in \omega$ and $\alpha = \alpha_0 0 \cdots 0 \alpha_m$, where all $\alpha_i \in W_1$ and not all of them empty.

(4) Let $\alpha = \alpha_0 0 \cdots 0 \alpha_m$ and $\beta = \beta_0 0 \cdots 0 \beta_n$, where all α_i, $\beta_j \in W_1$ and there are some $\alpha_i \neq \emptyset$ and $\beta_j \neq \emptyset$. Then

$$\begin{aligned} o(\alpha 0 \beta) &= o(\alpha_0 0 \cdots 0 \alpha_m 0 \beta_0 0 \cdots 0 \beta_n) \\ &= \omega^{o(\alpha_0^-)} + \cdots + \omega^{o(\alpha_m^-)} + \omega^{o(\beta_0^-)} + \cdots + \omega^{o(\beta_n^-)} \\ &= o(\alpha) + o(\beta). \end{aligned}$$

The proof is now complete. □

This lemma will be used tacitly in the following. Let β^{-n} be obtained from $\beta \in W_n$ by replacing every letter m with $m - n$. β^{+n} is similarly defined for $\beta \in W$. Below we write $o(\beta) < \omega$ for $o(\beta) = k$ for some $k \in \mathbb{N}$ and $o(\beta) \geq \omega$ otherwise.

Theorem 35 *$o(\alpha[\![k]\!]) = o(\alpha)[k]$ for all $\alpha \in W$.*

PROOF. There is nothing to prove if $\alpha = \emptyset$. If $\alpha = \beta 0$, then $\alpha[\![k]\!] = \beta$ and

$$(o(\alpha))[k] = (o(\beta) + 1)[k] = o(\beta) = o(\alpha[\![k]\!]).$$

Now let $\alpha = \beta m \gamma \langle n+1 \rangle$, where $m \leq n$ and $\gamma \in W_{n+1}$. There are four cases to consider.

(i) $\gamma = \emptyset$ and $\beta m = \emptyset$, i.e. $\alpha = \langle n+1 \rangle$ and $\alpha[\![k]\!] = n^{k+1}$:

$$o(\alpha)[k] = \omega_{n+1}[k] = \omega_n(k+1) = o(n^{k+1}) = o(\alpha[\![k]\!]).$$

(ii) $\gamma = \emptyset$ and $\beta m \neq \emptyset$. We use an induction on n:

(a) m is the minimum of the occurrences in βm.

$$\begin{aligned} o(\alpha) &= o(\beta m \langle n+1 \rangle) \\ &= \omega_m(o(\beta^{-m} 0 \langle n+1-m \rangle)) \\ &= \begin{cases} \omega_m(o(\beta^{-m}) + \omega_{n+1-m}) & , o(\beta^{-m}) \geq \omega, \\ \omega_m(o(\beta^{-m}) + 1 + \omega_{n+1-m}) & , o(\beta^{-m}) < \omega. \end{cases} \end{aligned}$$

And

$$o(\alpha)[k] = \begin{cases} \omega_m(o(\beta^{-m}) + \omega_{n-m}(k+1)) & , o(\beta^{-m}) \geq \omega,\ n > m, \\ \omega_m(o(\beta^{-m}) + k + 2) & , o(\beta^{-m}) \geq \omega,\ n = m, \\ \omega_m(o(\beta^{-m}) + 1 + \omega_{n-m}(k+1)) & , o(\beta^{-m}) < \omega. \end{cases}$$

On the other hand,

$$\begin{aligned} o(\alpha[\![k]\!]) &= o(\beta m n^{k+1}) \\ &= \omega_m(o(\beta^{-m} 0 \langle n-m \rangle^{k+1})) \\ &= \begin{cases} \omega_m(o(\beta^{-m}) + o(\langle n-m \rangle^{k+1})) & , o(\beta^{-m}) \geq \omega,\ n > m, \\ \omega_m(o(\beta^{-m}) + 1 + k + 1) & , o(\beta^{-m}) \geq \omega,\ n = m, \\ \omega_m(o(\beta^{-m}) + 1 + o(\langle n-m \rangle^{k+1})) & , o(\beta^{-m}) < \omega, \end{cases} \\ &= \begin{cases} \omega_m(o(\beta^{-m}) + \omega_{n-m}(k+1)) & , o(\beta^{-m}) \geq \omega,\ n > m, \\ \omega_m(o(\beta^{-m}) + k + 2) & , o(\beta^{-m}) \geq \omega,\ n = m, \\ \omega_m(o(\beta^{-m}) + 1 + \omega_{n-m}(k+1)) & , o(\beta^{-m}) < \omega. \end{cases} \end{aligned}$$

Note that the case $n = 0$ is also proved since m should then be 0.

(b) $n > 0$, $m > p$ and $\beta = \beta_0 p \cdots p \beta_{t+1}$, where all $\beta_i \in W_{p+1}$.

$$\begin{aligned} o(\alpha) &= o(\beta_0 p \cdots p \beta_{t+1} m \langle n+1 \rangle) \\ &= \omega_p(o(\beta_0^{-p} 0 \cdots 0 \beta_{t+1}^{-p}) \langle m-p \rangle \langle n+1-p \rangle) \\ &= \omega_p\big(\omega^{o(\beta_0^{-p-1})} + \cdots + \omega^{o(\beta_t^{-p-1})} + \omega^{o(\beta_{t+1}^{-p-1} \langle m-p-1 \rangle \langle n-p \rangle)}\big) \end{aligned}$$

Since $o(\beta_{t+1}^{-p-1}\langle m-p-1\rangle\langle n-p\rangle)$ is a limit, we have

$$\begin{aligned}
o(\alpha)[k] &= \omega_p(\omega^{o(\beta_0^{-p-1})} + \cdots + \omega^{o(\beta_t^{-p-1})} + \omega^{o(\beta_{t+1}^{-p-1}\langle m-p-1\rangle\langle n-p\rangle)[k]}) \\
&\overset{i.h.}{=} \omega_p(\omega^{o(\beta_0^{-p-1})} + \cdots + \omega^{o(\beta_t^{-p-1})} + \omega^{o((\beta_{t+1}^{-p-1}\langle m-p-1\rangle\langle n-p\rangle)[\![k]\!])}) \\
&= \omega_p(\omega^{o(\beta_0^{-p-1})} + \cdots + \omega^{o(\beta_t^{-p-1})} + \omega^{o(\beta_{t+1}^{-p-1}\langle m-p-1\rangle\langle n-p-1\rangle^{k+1})})
\end{aligned}$$

On the other hand,

$$\begin{aligned}
o(\alpha[\![k]\!]) &= o(\beta_0 p \cdots p\beta_{t+1} m n^{k+1}) \\
&= \omega_p(o(\beta_0^{-p} 0 \cdots 0 \beta_{t+1}^{-p})\langle m-p\rangle\langle n-p\rangle^{k+1}) \\
&= \omega_p(\omega^{o(\beta_0^{-p-1})} + \cdots + \omega^{o(\beta_t^{-p-1})} + \omega^{o(\beta_{t+1}^{-p-1}\langle m-p-1\rangle\langle n-p-1\rangle^{k+1})})
\end{aligned}$$

(iii) $\gamma \neq \emptyset$ and $\beta m = \emptyset$, i.e. $\alpha = \gamma\langle n+1\rangle$, $\gamma \in W_{n+1}$:

$$o(\alpha) = \omega_{n+1}(o(\gamma^{-n-1}0)) = \omega_{n+1}(o(\gamma^{-n-1}) + 1)$$

Since $o(\gamma^{-n-1}) > 0$, we have $o(\alpha)[k] = \omega_n(\omega^{o(\gamma^{-n-1})} \cdot (k+1) + 1)$. On the other hand,

$$\begin{aligned}
o(\alpha[\![k]\!]) &= o((\gamma n)^{k+1}) \\
&= \omega_n(o((\gamma^{-n}0)^{k+1})) \\
&= \omega_n(\omega^{o(\gamma^{-n-1})} \cdot (k+1) + 1).
\end{aligned}$$

(iv) $\gamma \neq \emptyset$ and $\beta m \neq \emptyset$. The claim will be shown by induction on n:

(a) m is the minimum of the occurrences in βm.

$$\begin{aligned}
o(\alpha) &= o(\beta m \gamma\langle n+1\rangle) \\
&= \omega_m(o(\beta^{-m} 0 \gamma^{-m}\langle n+1-m\rangle)) \\
&= \begin{cases} \omega_m(o(\beta^{-m}) + \omega_{n+1-m}(o(\gamma^{-n-1}0))) & , o(\beta^{-m}) \geq \omega, \\ \omega_m(o(\beta^{-m}) + 1 + \omega_{n+1-m}(o(\gamma^{-n-1}0))) & , o(\beta^{-m}) < \omega. \end{cases} \\
&= \begin{cases} \omega_m(o(\beta^{-m}) + \omega_{n+1-m}(o(\gamma^{-n-1}) + 1)) & , o(\beta^{-m}) \geq \omega, \\ \omega_m(o(\beta^{-m}) + 1 + \omega_{n+1-m}(o(\gamma^{-n-1}) + 1)) & , o(\beta^{-m}) < \omega. \end{cases}
\end{aligned}$$

Since $o(\gamma^{-n-1}) > 0$, we have

$$o(\alpha)[k] = \begin{cases} \omega_m(o(\beta^{-m}) + \omega_{n-m}(\omega^{o(\gamma^{-n-1})} \cdot (k+1) + 1)) & , o(\beta^{-m}) \geq \omega, \\ \omega_m(o(\beta^{-m}) + 1 + \omega_{n-m}(\omega^{o(\gamma^{-n-1})} \cdot (k+1) + 1)) & , o(\beta^{-m}) < \omega. \end{cases}$$

On the other hand,

$$
\begin{aligned}
o(\alpha[\![k]\!]) &= o(\beta m(\gamma n)^{k+1}) \\
&= \omega_m(o(\beta^m 0(\langle n-m\rangle\gamma^{-m})^{k+1})) \\
&= \begin{cases} \omega_m(o(\beta^{-m}) + o((\gamma^{-m})^{k+1}\langle n-m\rangle)) & , o(\beta^{-m}) \geq \omega, \\ \omega_m(o(\beta^{-m}) + 1 + o((\gamma^{-m})^{k+1}\langle n-m\rangle)) & , o(\beta^{-m}) < \omega, \end{cases} \\
&= \begin{cases} \omega_m(o(\beta^{-m}) + \omega_{n-m}(o((\gamma^{-n})^{k+1}0))) & , o(\beta^{-m}) \geq \omega, \\ \omega_m(o(\beta^{-m}) + 1 + \omega_{n-m}(o((\gamma^{-n})^{k+1}0))) & , o(\beta^{-m}) < \omega, \end{cases} \\
&= \begin{cases} \omega_m(o(\beta^{-m}) + \omega_{n-m}(\omega^{o(\gamma^{-n-1})}\cdot(k+1)+1)) & , o(\beta^{-m}) \geq \omega, \\ \omega_m(o(\beta^{-m}) + 1 + \omega_{n-m}(\omega^{o(\gamma^{-n-1})}\cdot(k+1)+1)) & , o(\beta^{-m}) < \omega. \end{cases}
\end{aligned}
$$

Note that the case $n = 0$ is also proved since m should then be 0.

(b) $n > 0$, $m > p$ and $\beta = \beta_0 p \cdots p\beta_{t+1}$, where all $\beta_i \in W_{p+1}$.

$$
\begin{aligned}
o(\alpha) &= o(\beta_0 p \cdots p\beta_{t+1} m\gamma\langle n+1\rangle) \\
&= \omega_p(o(\beta_0^{-p}0\cdots 0\beta_{t+1}^{-p}\langle m-p\rangle\gamma^{-p}\langle n+1-p\rangle)) \\
&= \omega_p\big(\omega^{o(\beta_0^{-p-1})} + \cdots + \omega^{o(\beta_t^{-p-1})} + \omega^{o(\beta_{t+1}^{-p-1}\langle m-p-1\rangle\gamma^{-p-1}\langle n-p\rangle)}\big)
\end{aligned}
$$

Since $o(\beta_{t+1}^{-p-1}\langle m-p-1\rangle\gamma^{-p-1}\langle n-p\rangle)$ is a limit, we have

$$
\begin{aligned}
o(\alpha)[k] &= \omega_p\big(\omega^{o(\beta_0^{-p-1})} + \cdots + \omega^{o(\beta_t^{-p-1})} + \omega^{o(\beta_{t+1}^{-p-1}\langle m-p-1\rangle\gamma^{-p-1}\langle n-p\rangle)[k]}\big) \\
&\overset{i.h.}{=} \omega_p\big(\omega^{o(\beta_0^{-p-1})} + \cdots + \omega^{o(\beta_t^{-p-1})} + \omega^{o((\beta_{t+1}^{-p-1}\langle m-p-1\rangle\gamma^{-p-1}\langle n-p\rangle)[\![k]\!])}\big) \\
&= \omega_p\big(\omega^{o(\beta_0^{-p-1})} + \cdots + \omega^{o(\beta_t^{-p-1})} + \omega^{o(\beta_{t+1}^{-p-1}\langle m-p-1\rangle(\gamma^{-p-1}\langle n-p-1\rangle)^{k+1})}\big)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
o(\alpha[\![k]\!]) &= o(\beta_0 p \cdots p\beta_{t+1} m(\gamma n)^{k+1}) \\
&= \omega_p(o(\beta_0^{-p}0\cdots 0\beta_{t+1}^{-p}\langle m-p\rangle(\gamma^{-p}\langle n-p\rangle)^{k+1})) \\
&= \omega_p\big(\omega^{o(\beta_0^{-p-1})} + \cdots + \omega^{o(\beta_t^{-p-1})} + \omega^{o(\beta_{t+1}^{-p-1}\langle m-p-1\rangle(\gamma^{-p-1}\langle n-p-1\rangle)^{k+1})}\big)
\end{aligned}
$$

This completes the proof. $\square$

Remember that an ordinal $\alpha = \omega^{\alpha_0} + \cdots + \omega^{\alpha_n}$ below ε_0 is said to be in Cantor normal form if $\alpha > \alpha_0 \geq \cdots \geq \alpha_n$. We can demand the same property from every α_i and so on. Furthermore, if we make no difference between an ordinal below ε_0 in Cantor normal form and an tree-ordinal which has the same tree figure, we can specify a set $\mathbb{B}$ of all tree-ordinals in so-called Cantor normal form. This implies in turn that $\mathbb{B}$ corresponds isomorphically to the set $NF \subseteq S$ of all words in normal forms.

Let $NF(W) \subseteq W$ be the set of all worms which are converses of a word in NF. The worms in $NF(W)$ are also said to be in Cantor normal form and the set $NF(W)$ is isomorphic to ε_0.

Lemma 36 *$NF(W)$ can be characterized inductively as follows:*

(1) $\emptyset$ and any worm of length 1 belong to $NF(W)$;
(2) assume that the length of the worm α is larger than 1 and $\alpha = \alpha_0 0 \cdots 0 \alpha_n$, where all $\alpha_i \in W_1$. Then $\alpha \in NF(W)$ iff all $\alpha_i^- \in NF(W)$ and $o(\alpha_{j+1}^-) \leq o(\alpha_j^-)$ for all $j < n$.

Note that $o(\alpha) \in \mathbb{B}$ for every $\alpha \in NF(W)$, so we might talk about the linear ordering $<$ of ordinals. It is also obvious that $\alpha[\![k]\!] \in NF(W)$ for all $\alpha \in NF(W)$ and $k \in \mathbb{N}$. Let $\prec_0$ be the well-ordering on $NF(W)$ induced by the isomorphism o.

Lemma 37 *$o \restriction NF(W) \colon NF(W) \to \mathbb{B}$ is an order-preserving isomorphism.*

Having established an correspondence between W and $\mathbb{T}$ (or between $NF(W)$ and $\mathbb{B}$) it is now obvious that the Worm principle is the counterpart of the Hydra battle game on the tree-ordinals in $\mathbb{T}$ (resp. on the ordinals up to ε_0).

On the other hand, the Hydra battle game has a direct connection to the Hardy-Wainer hierarchy. It is a folklore that the Hardy-Wainer hierarchy up to ε_0 features exactly the provably recursive functions in PA. Fairtlough and Wainer [7] showed that an similar characterization of provably recursive functions in PA is possible by using the tree-ordinals from $\mathbb{T}$. Furthermore, Weiermann [32] made an refinement in such a way that how fast heads of a hydra should be multiplied at cutting off the right-most head, so that the Hydra battle game on the ordinals up to ε_0 remains unprovable in PA. Using the same idea we show that an analogous process is possible with respect to the tree-ordinals from $\mathbb{T}$.

First we recall some well-known definitions and lemmata from subrecursive hierarchy theory based on the fundamental sequences defined in Definition 30. Let f, g range over unary arithmetical functions, k, n, x over $\mathbb{N}$, and α, β, λ, etc. over $\mathbb{T}$.

Definition 38 *Let $\lambda \in Lim$.*

(1) $P_x^f 0 := 0$, $P_x^f(\alpha+1) := \alpha$ and $P_x^f \lambda := P_x^f(\lambda[f(x)])$.
(2) $Q_x^f 0 := 0$, $Q_x^f(\alpha+1) := \alpha$ and $Q_x^f \lambda := \lambda[f(x)]$.
(3) Let $R \in \{P, Q\}$.
- *$R_x \alpha := R_x^{id} \alpha$.*
- *$\alpha \succ_f^{R,n} \beta$ if $\beta = R_n^f \cdots R_1^f \alpha$.*
- *$\alpha \succ_f^R \beta$ if $\beta = R_n^f \cdots R_1^f \alpha$ for some positive n.*

- $\alpha \succ_k^R \beta$ *if* $\alpha \succ_f^R \beta$*, where* $f \equiv k$*.*
- $\alpha \succeq_k^R \beta$ *if* $\alpha \succ_k^R \beta$ *or* $\alpha = \beta$*.*

(4) $G_x(0) := 0$*,* $G_x(\alpha+1) = G_x(\alpha)+1$ *and* $G_x(\lambda) := G_x(\lambda[x])$*.*
(5) $H_0^f(x) := x$*,* $H_{\alpha+1}^f(x) := H_\alpha^f(x+1)$ *and* $H_\lambda^f(x) := H_{\lambda[f(x)]}^f(x)$*.*
(6) $H_\alpha := H_\alpha^{id}$*.*
(7) $\mathrm{mc}(m) := m$ *and* $\mathrm{mc}(\alpha) := \max\{m_1, \dots, m_n, \mathrm{mc}(\alpha_1), \dots, \mathrm{mc}(\alpha_n)\}$*, where* $\alpha = \omega^{\alpha_1} \cdot m_1 + \cdots + \omega^{\alpha_n} \cdot m_n$ *such that* $\alpha_i > \alpha_{i+1}$ *for each* $i < n$*.*
(8) $\alpha_0(\beta) := \beta$*,* $\alpha_{n+1}(\beta) := \alpha^{\alpha_n(\beta)}$ *and* $\alpha_n := \alpha_n(1)$*.*
(9) $\varepsilon_0 := \langle \omega_{n+1} \rangle_n$ *and* $\varepsilon_0[k] := \omega_{k+1}$*.*
(10) $H_{\varepsilon_0}(x) := H_{\varepsilon_0[x]}(x)$*.*

Note that $G_1(\omega_{n+1}) \geq 2_n(2)$.

Theorem 39 *Let* α*,* $\beta \in \mathbb{T}$*.*

(1) G_α *is increasing (strictly if* α *infinite), and if* $\beta \prec \alpha[n]$*, then* $G_\beta(n) < G_\alpha(n)$ *for all* n *and* G_α *eventually dominates* G_β*.*
(2) H_α *is strictly increasing, and if* $\beta \prec \alpha[n]$*, then* $H_\beta(n) < H_\alpha(n)$ *for all* n *and* H_α *eventually dominates* H_β*.*
(3) H_α *is provably recursive in* PA*.*
(4) Every provably recursive function in PA *is dominated by* H_α *for some* α*.*
(5) H_{ε_0} *is not provably recursive in* PA*.*

PROOF. See Fairtlough and Wainer [7]. □

Given $R \in \{P, Q\}$ set $R_x^{(0)}\alpha := \alpha$ and $R_x^{(i+1)}\alpha := R_x R_x^{(i)}\alpha$. For $\alpha \in \mathbb{T}$ and $n \in \mathbb{N}$ let $\alpha[\omega := n]$ be the natural number obtained by replacing every occurrence of ω in α with n.

Lemma 40 *Let* $R \in \{P, Q\}$*.*

(1) $R_x(\alpha+\beta) = \alpha + R_x\beta$ *for* $\beta \neq 0$*.*
(2) If $\alpha \succ_x^P \beta$ *then* $\alpha \succ_x^Q \beta$*.*
(3) If $\alpha \succ_x^R \beta$ *then* $\gamma + \alpha \succ_x^R \gamma + \beta$*.*
(4) If $\alpha \succ_x^R \beta$ *then* $\omega^\alpha \succ_x^R \omega^\beta$*.*
(5) If λ *is a limit then* $\lambda[x+1] \succ_0^Q \lambda[x]$*.*
(6) If λ *is a limit then* $\lambda[x+1] \succ_{x+1}^Q \lambda[x]+1$*.*
(7) If $x > 0$ *then* $\omega^{\alpha+1} \succ_x^Q \omega^\alpha + \omega^\alpha$*.*
(8) If $x \geq 0$ *then* $\omega^{\alpha+1} \succ_x^Q \omega^\alpha + 1$*.*
(9) If $x > 0$ *then* $\omega_{n+1}(\alpha+1) \succ_x^Q \omega_{n+1}(\alpha) + \omega_{n+1}$*.*
(10) If $\alpha > 0$ *then* $\alpha \succeq_{x+1}^Q P_x\alpha + 1$*.*
(11) $\alpha \succ_x^Q P_x\alpha$*.*
(12) If f*,* g *are increasing, where* $g(i) \leq f(i)$ *for all* i*, and* $\alpha \succ_g^{R,m} \beta$*, then* $\alpha \succ_f^{R,n} \beta$ *for some* $n \geq m$*.*

(13) If $\alpha \succeq_x^Q \beta \succ_x^{P,m} \gamma$ then $\alpha \succ_x^{P,n} \gamma$ for some $n \geq m$.
(14) There are at most $G_{x+1}(\alpha)$ elements in $\{\beta \prec \alpha \colon \mathrm{mc}(\beta) \leq x+1\}$.
(15) $\alpha[\omega := x+1] \leq G_x(\alpha) \leq \alpha[\omega := x+2]$.
(16) $G_x(\alpha) = \min\{i \colon P_x^{(i)}\alpha = 0\}$.
(17) $H_\alpha(x) = \min\{i \colon P_{i+x-1} \cdots P_x\alpha = 0\} + x$, so

$$H_\alpha(x) = \min\{i \geq x \colon P_i \cdots P_x\alpha = 0\} + 1.$$

PROOF. (1) $\sim$ (16) are more or less obvious. (17) is proved in Fairtlough and Wainer [7]. $\square$

Lemma 41 *Given $n \in \mathbb{N}$ set $g_n(i) := |i|_n$. Set also $\beta := \omega_{n+1}(\lambda) + \omega_{n+1}$ where $\lambda \in \mathbb{T}$ is a limit. Then there exists an $i \geq H_\lambda(1)$ such that $\beta \succ_{g_n}^{P,i} \omega_{n+1}(0)$.*

PROOF. Let $L := H_\lambda(1) - 1 = \min\{i \colon P_i \cdots P_1\lambda = 0\}$. By definition we have $g_n(i) \geq 1$ for all i. Further we obtain

$$\begin{aligned} \beta &= \omega_{n+1}(\lambda) + \omega_{n+1} \\ &\succ_1^P \omega_{n+1}(\lambda) + P_1\omega_{n+1} \\ &\succ_1^P \omega_{n+1}(\lambda) + P_1P_1\omega_{n+1} \\ &\succ_1^P \cdots \end{aligned}$$

Hence, there exists $i_0 \geq 2_n(2)$ such that $\beta \succ_1^{P,i_0} \omega_{n+1}(\lambda)$ since

$$\min\{n \colon P_1^{(n)}\omega_{n+1} = 0\} = G_1\omega_{n+1} \geq 2_n(2).$$

And for $i \geq i_0$ we have $2 \leq g_n(i)$. In addition, we have

$$\begin{aligned} \omega_{n+1}(\lambda) &\succ_2^P \omega_{n+1}(P_1\lambda + 1) \\ &\succ_2^Q \omega_{n+1}(P_1\lambda) + \omega_{n+1} \\ &\succ_2^P \omega_{n+1}(P_1\lambda) + P_2\omega_{n+1} \\ &\succ_2^P \omega_{n+1}(P_1\lambda) + P_2P_2\omega_{n+1} \\ &\succ_2^P \cdots \end{aligned}$$

Therefore, there exists $i_0 \geq 2_n(2)$ such that $\beta \succ_1^{P,i_0} \omega_{n+1}(\lambda)$ since

$$\min\{k \colon P_2^{(k)}\omega_{n+1} = 0\} = G_2\omega_{n+1} \geq 3_n(3).$$

This process shows that given $k \leq L$ there is a sequence $\langle i_\ell \rangle_{\ell \leq k}$ such that

$$i_\ell \geq (\ell+2)_n(\ell+2)$$

for all $\ell \leq k$ and $\beta \succ_{g_n}^{P,i_0+i_1+\cdots+i_k} \omega_{n+1}(P_k \cdots P_1\lambda)$. The claim follows now from the fact that $i_0 + \cdots + i_L \geq L+1 = H_\lambda(1)$. $\square$

Lemma 42 *Given $\alpha \in \mathbb{T} \cup \{\varepsilon_0\}$ set $f_\alpha(i) := |i|_{H_\alpha^{-1}(i)}$. Then for every $\alpha := \omega_{n+1}(\omega_n) + \omega_{n+1}$, $n \geq 2$, there is a $\delta \geq \omega_{n+1}(0)$ such that $\alpha \succ^{P,i}_{f_{\varepsilon_0}} \delta$ for some $i \geq H_{\omega_n}(1)$.*

PROOF. If $k \leq H_{\omega_n}(1) =: i_0$, then $H_{\varepsilon_0}^{-1}(k) \leq H_{\varepsilon_0}^{-1}(i_0) \leq n$. Hence $f_{\varepsilon_0}(k) = |k|_{H_{\varepsilon_0}^{-1}(k)} \geq |k|_n = g_n(k)$. By Lemma 41 there exists $\delta > \omega_{n+1}(0)$ such that $\alpha \succ^{P,i_0}_{g_n} \delta$. This implies that there is $i \geq i_0$ such that $\alpha \succ^{P,i}_{f_{\varepsilon_0}} \delta$. $\square$

Theorem 43 *Let $n \in \mathbb{N}$.*

(1) Let $g_n(i) := |i|_n$. Then $\mathrm{PA} \nvdash \forall k \, \exists m \, Q^{g_n}_m \cdots Q^{g_n}_1 \omega_k = 0$.
(2) Let $f(i) := |i|_{H_{\varepsilon_0}^{-1}(i)}$. Then $\mathrm{PA} \nvdash \forall k \, \exists m \, Q^{f}_m \cdots Q^{f}_1 \omega_k = 0$.

PROOF. It follows from the fact that the function $i \mapsto H_{\omega_i}(1)$ is not provably recursive in PA. Cf. Fairtlough and Wainer [7]. $\square$

The expected counterpart, i.e. provability, can be shown as follows.

Theorem 44 *For $\alpha \in \mathbb{T}$ let $f_\alpha(i) := |i|_{H_\alpha^{-1}(i)}$. Then*

$$\mathrm{PRA} \vdash \forall k \, \exists m \, Q^{f_\alpha}_m \cdots Q^{f_\alpha}_1 \omega_k = 0.$$

PROOF. Assume that k is large enough. How large k should be will be obvious from the context. We claim that $Q^{f_\alpha}_m \cdots Q^{f_\alpha}_1 \omega_k = 0$, where $m := 2_{H_{\omega^\alpha \cdot 2}(k)}$. Assume otherwise. Since

$$\mathrm{mc}(\omega_k[f_\alpha(1)] \cdots [f_\alpha(i)]) \leq f_\alpha(i) + 2 \leq f_\alpha(m) + 2$$

for every $i \leq m$ we have by Lemma 40.(14) $m \leq G_{2+f_\alpha(m)}(\omega_k)$. By Lemma 40.(15)

$$\begin{aligned} m &\leq (4 + |2_{H_{\omega^\alpha \cdot 2}(k)}|_{H_\alpha^{-1}(2_{H_{\omega^\alpha \cdot 2}(k)})})_k(1) \\ &\leq (4 + |2_{H_{\omega^\alpha \cdot 2}(k)}|_{H_{\omega^\alpha}(k)})_k(1) \\ &= (5 + 2_{H_{\omega^\alpha \cdot 2}(k) - H_{\omega^\alpha}(k)})_k(1) \\ &< 2_{H_{\omega^\alpha \cdot 2}(k)} \;=\; m \end{aligned}$$

for sufficiently large k. Contradiction! $\square$

Note that, for any α, $\beta \in \mathbb{T}$, there is a $k \in \mathbb{N}$ such that

$$\min\{m \colon Q^{f_\alpha}_m \cdots Q^{f_\alpha}_1 \beta = 0\} \leq \min\{m \colon Q^{f_\alpha}_m \cdots Q^{f_\alpha}_1 \omega_k = 0\}.$$

The existence of such a k is PA-provable.

Theorem 45 *Let $\alpha \in \mathbb{T} \cup \{\varepsilon_0\}$.*

(1) $\mathbf{EWD}(\mathrm{inv})$ *is* PA*-provable.*
(2) $\mathbf{EWD}(g_n)$ *is not* PA*-provable for* $g_n(i) := |\cdot|_n$*,* $n \in \mathbb{N}$*.*
(3) Let $f_\alpha(i) := |\cdot|_{H_\alpha^{-1}(\cdot)}$ *. Then* $\mathbf{EWD}(f_\alpha)$ *is* PA*-provable iff* $\alpha \in \mathbb{T}$*.*

PROOF. Obvious by Theorem 35, Theorem 43, and Theorem 44. □

It is now obvious to see that the same results hold for the worms in normal form. Define

$$\mathbf{EWD}_{nf}(f) :\equiv \forall \alpha \in NF(W)\, \exists n\, (\alpha(f, n) = \emptyset).$$

Theorem 46 *Let $\alpha \leq \varepsilon_0$ be an ordinal.*

(1) $\mathbf{EWD}_{nf}(\mathrm{inv})$ *is* PA*-provable.*
(2) $\mathbf{EWD}_{nf}(g_n)$ *is not* PA*-provable for* $g_n(i) := |\cdot|_n$*,* $n \in \mathbb{N}$*.*
(3) Let $f_\alpha(i) := |\cdot|_{H_\alpha^{-1}(\cdot)}$ *. Then* $\mathbf{EWD}_{nf}(f_\alpha)$ *is* PA*-provable iff* $\alpha < \varepsilon_0$*.*

3.3 Schütte-Simpson's ordinal notation system

Another interesting ordinal notation system for ε_0 is introduced by Schütte and Simpson [33]. It is called $\pi_0(\omega)$ and a segment of $\pi(\omega)$ defined by letting out the addition and the function $\alpha \mapsto \omega^\alpha$ in the construction of the ordinal notation system developed by Buchholz [6].

The new defined ordinal terms seem, at least for the author, so artificial that it would make no sense to say more about them other than their combinatorial property. Hence it is all the more meaningful to see that there is a canonical correspondence between them and worms.

In the following, we will proceed at first as in [33]. However with different access to the resulting notation system, i.e. we don't refer to the original collapsing functions any more. This seems to be somewhat more technical, but has the advantage that one can easily see the correspondence between Schütte-Simpson's system and Beklemishev's one.

In this section the small Greek letters α, β, $\gamma, \ldots$ range over ordinals. We set $\Omega_0 := 0$ and, for $i > 0$, Ω_i the i-th infinite regular ordinal and $\Omega_\omega := \sup\{\Omega_i \colon i < \omega\}$.

Definition 47 *We define $B_i^m(\alpha)$, $B_i(\alpha)$ and $\pi_i(\alpha)$ (by the main induction on α and the subsidiary induction on m):*

$(B1)$ *if* $\gamma = 0$ *or* $\gamma < \Omega_i$, *then* $\gamma \in B_i^m(\alpha)$;
$(B2)$ *if* $i \leq j$, $\beta < \alpha$, $\beta \in B_j(\beta)$, *and* $\beta \in B_i^m(\alpha)$, *then* $\pi_j\beta \in B_i^{m+1}(\alpha)$;
$(B3)$ $B_i(\alpha) := \cup\{B_i^m(\alpha)\colon m < \omega\}$;
$(B4)$ $\pi_i\alpha := \min\{\eta\colon \eta \notin B_i(\alpha)\}$.

Lemma 48 (Schütte and Simpson [33])

(1) *If* $k < m$, *then* $B_i^k(\alpha) \subseteq B_i^m(\alpha)$.
(2) *If* $i \leq j$ *and* $\alpha \leq \beta$, *then* $B_i(\alpha) \subseteq B_j(\beta)$, $\pi_i\alpha \leq \pi_j\beta$.
(3) $\Omega_i \leq \pi_i\alpha < \Omega_{i+1}$.
(4) *If* $\gamma \in B_i(\alpha)$ *and* $\gamma < \Omega_{i+1}$, *then* $\gamma < \pi_i\alpha$.
(5) *If* $\alpha \in B_i(\alpha)$ *and* $\alpha < \beta$, *then* $\pi_i\alpha < \pi_i\beta$.
(6) *If* $\alpha \in B_i(\alpha)$, $\beta \in B_i(\beta)$, *and* $\pi_i\alpha = \pi_i\beta$, *then* $\alpha = \beta$.

Definition 49 $\pi(\omega)$ *is inductively defined as follows:*

(1) $0 \in \pi(\omega)$;
(2) *if* $\alpha \in \pi(\omega)$ *and* $\alpha \in B_i(\alpha)$ *then* $\pi_i\alpha \in \pi(\omega)$.

To see that $\pi(\omega)$ is a primitive recursive set we must be able to decide the relation $\alpha \in B_i(\alpha)$ for $\alpha \in \pi(\omega)$. For this we introduce an auxiliary concept of coefficients sets. The idea stems from Rathjen and Weiermann [34].

Definition 50 *Inductive definition of a set of ordinals* $K_i\alpha$ *for* $\alpha \in \pi(\omega)$.

(1) $K_i(0) := \emptyset$;
(2) $K_i(\pi_j\alpha) := \begin{cases} \{\alpha\} \cup K_i(\alpha) & \text{if } i \leq j, \\ \emptyset & \text{otherwise.} \end{cases}$

The following lemma can be shown by an simple induction.

Lemma 51 *Let* $\alpha \in \pi(\omega)$. *Then* $K_i(\alpha) < \beta$ *iff* $\alpha \in B_i(\beta)$.

Theorem 52 *Let* α, $\beta \in \pi(\omega)$.

(1) *The set* $\pi(\omega)$ *is primitive recursive and can be characterized as follows:*
- $0 \in \pi(\omega)$;
- *if* $\alpha \in \pi(\omega)$ *and* $K_i(\alpha) < \alpha$, *then* $\pi_i\alpha \in \pi(\omega)$.

(2) $\pi(\omega) = B_0(\Omega_\omega)$.
(3) $\alpha < \beta$ *if one of the following three cases holds:*
- $\alpha = 0$ *and* $\beta \neq 0$;
- $\alpha = \pi_i\delta$, $\beta = \pi_j\gamma$, *and* $i < j$;
- $\alpha = \pi_i\delta$, $\beta = \pi_i\gamma$, *and* $\delta < \gamma$.

PROOF. (1) and (2) are obvious. (3) follows from Lemma 48. □

Now it can be decided primitive recursively whether $\alpha < \beta$, $\alpha = \beta$, or $\alpha > \beta$ for any α, $\beta \in \pi(\omega)$. In other words, $\alpha < \beta$ can be read as a Δ_0-formula.

Definition 53 *(1) We consider every element of $\pi(\omega)$ as a term defined according to the induction and call it an* ordinal term.

(2) $\pi_0(\omega)$ is the set of ordinals from $\pi(\omega)$ which are less than Ω_1. That is,

$$\pi_0(\omega) := \{\alpha \in \pi(\omega) \mid \alpha = 0 \text{ or } \alpha = \pi_0\beta \text{ for some } \beta \in \pi(\omega)\} = \pi_0\Omega_\omega.$$

If we use the following abbreviations

$$i_1 \cdots i_k 0 := \pi_{i_1} \cdots \pi_{i_k} 0,$$

then every $\alpha \in \pi_0(\omega)$ is of the form $\alpha = 0\alpha_1 0 \cdots 0\alpha_n 0^m 0$ for some n, $m \in \mathbb{N}$, where $\alpha_i \in W_1$. Note that, if $n = 0$, then $\alpha = 0^m 0$, hence $\alpha = 0$ if $m = 0$. The following lemma reveals something about the relationship between the elements of $\pi_0(\omega)$ and $NF(W)$.

Lemma 54 *Let $\alpha = 0\alpha_1 0 \cdots 0\alpha_n 0^m 0$ be in $\pi_0(\omega)$ with $n > 1$. If $\alpha_i = \emptyset$ for some i, $1 \leq i < n$, then $\alpha_{i+1} = \emptyset$.*

PROOF. Assume $\alpha_i = \emptyset$ and $\alpha_{i+1} \neq \emptyset$ for some $i < n$. Then α has the form $\pi_0 \cdots \pi_0 \pi_0 \pi_l \cdots 0$ for some $\ell > 0$. However, this cannot be in $\pi_0(\omega)$, since

$$K_0(\pi_0\pi_\ell \cdots 0) = \{\pi_\ell \cdots , \dots\} \not< \pi_0\pi_\ell \cdots 0.$$

Hence $\alpha_{i+1} = \emptyset$. $\square$

From now on we may assume for every $\alpha \in \pi_0(\omega)$ that α is of the form $\alpha = 0\alpha_1 0 \cdots 0\alpha_n 0^m 0$, where $\alpha_i \in W_1 \setminus \{\emptyset\}$ if $n \geq 1$.

Definition 55 *(1) By a* functional *we mean a finite sequence γ of natural numbers such that $\gamma 0 \in \pi(\omega)$.*

(2) For $\alpha = i_1 \cdots i_n 0 \in \pi(\omega)$, $n \geq 1$, define a functional $\bar{\alpha}$ by

$$\bar{\alpha} := \langle i_1 + 1 \rangle \cdots \langle i_n + 1 \rangle.$$

The following lemma can be proved by a simple induction.

Lemma 56 *Let α, $\beta > 0$ and $\gamma, \delta \in \pi_0(\omega)$.*

(1) $\bar{\alpha}\gamma \in \pi(\omega) \setminus \pi_0(\omega)$.
(2) $\bar{\alpha}\gamma < \bar{\beta}\delta$ iff $\alpha < \beta$, or $\alpha = \beta$ and $\gamma < \delta$.
(3) $K_{i+1}(\bar{\alpha}\gamma) < \bar{\beta}\gamma$ iff $K_i(\alpha) < \beta$.

Lemma 57 *For every $\gamma \in \pi(\omega) \setminus \pi_0(\omega)$ there are uniquely determined $\alpha > 0$ and $\delta \in \pi_0(\omega)$ such that $\gamma = \bar{\alpha}\delta$. In fact, if $\gamma = \gamma_1 0 \gamma_2$ with $\gamma_1 \in W_1$, then $\gamma = \bar{\alpha} 0 \gamma_2$, where $\alpha := \gamma_1^{-} 0$.*

PROOF. The uniqueness of α and δ follows from Lemma 56.(2). It remains to show that $\alpha := \gamma_1^{-} 0 \in \pi(\omega)$. We use the induction on the length of γ_1.

- $\gamma_1 = \langle i+1 \rangle$. Then $\alpha = \pi_i 0$ is obviously in $\pi_0(\omega)$.
- $\gamma_1 = \langle i+1 \rangle \eta$, where $\eta 0 \gamma_2 \in \pi(\omega) \setminus \pi_0(\omega)$ and $K_{i+1}(\eta) < \eta$. Then by I.H. $\beta := \eta^{-} 0 \in \pi(\omega)$ with $\eta = \bar{\beta} 0 \gamma_2$, and $K_i(\beta) < \beta$ follows from $K_{i+1}(\eta) < \eta$. So $\alpha = i\beta \in \pi(\omega)$. □

Lemma 58 *Let α, $\beta > 0$ and $\delta \in \pi_0(\omega)$. If $K_0(\bar{\alpha}\delta) < \bar{\beta}\delta$, then $K_0(\alpha) < \beta$.*

PROOF. By induction on α. If $\alpha = \pi_i 0$, then it is obvious since $\beta > 0$. Now let $\alpha = \pi_i \eta$ and $\eta > 0$. Then $\bar{\alpha}\delta = \pi_{i+1}\bar{\eta}\delta$. Since $K_0(\bar{\alpha}\delta) < \bar{\beta}\delta$ we have $\bar{\eta}\delta < \bar{\beta}\delta$ and $K_0(\bar{\eta}\delta) < \bar{\beta}\delta$. By I.H. $K_0(\eta) < \beta$, hence $K_0(\pi_i \eta) < \beta$. □

Lemma 59 *Let α, $\beta > 0$ such that $K_0(\alpha) < \beta$ and $\delta \in \pi_0(\omega)$. Then*

$$K_0(\bar{\alpha}\delta) < \bar{\beta}\delta \quad \textit{iff} \quad \delta < \pi_0 \bar{\beta}\delta$$

PROOF. If $K_0(\bar{\alpha}\delta) < \bar{\beta}\delta$, then $K_0(\delta) < \bar{\beta}\delta$. Hence $\delta \in B_0(\bar{\beta}\delta)$. By Theorem 48.(4) we have $\delta < \pi_0\bar{\beta}\delta$. Now assume $\delta < \pi_0\bar{\beta}\delta$, i.e., $K_0(\delta) < \bar{\beta}\delta$. There are two cases.

- If $\alpha = \pi_0 0$, then $\bar{\alpha}\delta = \pi_{i+1}\delta$. Hence $K_0(\bar{\alpha}\delta) < \bar{\beta}\delta$.
- Let $\alpha = \pi_i \eta$, $\eta > 0$, and $\bar{\alpha}\delta = \pi_{i+1}\bar{\eta}\delta$. Since $K_0(\alpha) < \beta$ then $\eta < \beta$ and $K_0(\eta) < \beta$. Hence $\bar{\eta}\delta < \bar{\beta}\delta$ and $K_0(\bar{\eta}\delta) < \bar{\beta}\delta$ by I.H. □

Definition 60 *Define $[\alpha]$ for $\alpha \in \pi_0(\omega)$ as follows.*

(1) $[0] := 0$*;*
(2) if $\alpha = 0^{m+1}0$ for some m, then $[\alpha] := 0\bar{\alpha} = 01^{m+1}$;
(3) if $\alpha = 0\beta$ with $\beta \in \pi(\omega) \setminus \pi_0(\omega)$, then $[\alpha] := 0\bar{\beta}$.

At first glance this definition seems to be somewhat different from the original one in [33]. But it isn't because of Lemma 54.

Lemma 61 *If α, $\delta \in \pi_0(\omega)$, then $[\alpha]\delta \in \pi_0(\omega)$ iff $\delta < [\alpha]\delta$.*

PROOF. By induction on α. If $\alpha = 0$, then $[\alpha]\delta = \pi_0\delta$. Hence $[\alpha]\delta \in \pi_0(\omega)$ iff $K_0(\delta) < \delta$. This is exactly the case if $\delta < \pi_0\delta = [\alpha]\delta$ because of Lemma 54.

Let $\alpha = \pi_0\beta$, $K_0(\beta) < \beta$, and $[\alpha]\delta = \pi_0\bar{\gamma}\delta$, where $\alpha = \gamma = 0^{m+1}0$ for some m by Lemma 54 if $\beta \in \pi_0(\omega)$, and $\gamma = \beta$ otherwise. In the first case, $\beta < \alpha$, hence $K_0(\beta) < \alpha$ and $K_0(\alpha) < \alpha$. Therefore, we have $K_0(\gamma) < \gamma$ in both cases. By Lemma 56.(1) $\bar{\gamma}\delta \in \pi(\omega)$. Moreover, by Lemma 59 $K_0(\bar{\gamma}\delta) < \bar{\gamma}\delta$ iff $\delta < \pi_0\bar{\gamma}\delta = [\alpha]\delta$. □

Corollary 62 $[\alpha]$ *is a functional for every* $\pi_0(\omega)$.

The following characterization is obvious.

Lemma 63 *Let* α, β, γ, δ, $[\alpha]\gamma$ *and* $[\beta]\delta$ *be from* $\pi_0(\omega)$. *Then* $[\alpha]\gamma < [\beta]\delta$ *in exactly one of the following two cases:*

$$\alpha < \beta \quad \text{or} \quad (\alpha = \beta \text{ and } \gamma < \delta)$$

Note that if $\alpha = 0\alpha_1 0 \cdots 0\alpha_n 0^m 0 \in \pi_0(\omega)$ and $\alpha_i = 1\eta$, then $\eta = 1^k$ for some k. If not, we would have $K_0(\alpha_i 0 \cdots 0\alpha_n 0^m 0) \not< \alpha_i 0 \cdots 0\alpha_n 0^m 0$ which is not allowed. Hence the following lemma makes sense.

Lemma 64 *For every* $\gamma \in \pi_0(\omega) \setminus \{0\}$ *there are unique* α, $\eta \in \pi_0(\omega)$ *such that* $\gamma = [\alpha]\eta$. *In fact, if* $\gamma = 0\beta 0\delta$ *and* $\beta \in W_1$, *then*

$$\gamma = \begin{cases} [0]0\delta & \text{if } \beta = \emptyset, \\ [\beta']0\delta & \text{otherwise,} \end{cases}$$

where $\beta' := \begin{cases} \beta^- 0 & \text{if } \beta = 1^k \text{ for some } k, \\ 0\beta^- 0 & \text{if } \beta = j\eta \text{ for some } \eta \text{ and } j \geq 2. \end{cases}$

PROOF. The uniqueness follows from Lemma 63. Let

$$\gamma' := \begin{cases} 0 & \text{if } \beta = \emptyset, \\ \beta' & \text{otherwise.} \end{cases}$$

We claim $\gamma' \in \pi_0(\omega)$ and $\gamma = [\gamma']0\delta$. In case of $\beta = \emptyset$ it is obvious. Let $\beta \neq \emptyset$. Since $K_0(\beta 0\delta) < \beta 0\delta$, we have $K_0(0\delta) < \beta 0\delta$ and $0\delta < 0\beta 0\delta = \gamma$. And by Lemma 57 and Lemma 58 $\beta^- 0 \in \pi(\omega)$ and $K_0(\beta^- 0) < \beta^- 0$. Hence it holds that $0\beta^- 0 \in \pi_0(\omega)$. If $\beta = 1\eta$ for some η, then $\beta^- 0 = 0^m$ for some m. So $[\beta'] = [\beta^- 0] = 0\beta^{-+} = 0\beta$. If $\beta = j\eta$ for some η and $j \geq 2$, then $[\beta'] = [0\beta' 0] = 0\beta^{-+} = 0\beta$. □

Lemma 65 *Let* $n > 0$ *and* $\alpha = 0\alpha_1 0 \cdots 0\alpha_n 0^m 0$ *from* $\pi_0(\omega)$ *with a non-empty* α_n. *Then* $\alpha'_1 \geq \cdots \geq \alpha'_n$ *and* $\alpha = [\alpha'_1] \cdots [\alpha'_n] 0^m 0$.

PROOF. The second claim is true by Lemma 64. For the first one, note that for every $i < n$

$$[\alpha'_{i+1}] \cdots [\alpha'_n]0^m 0 < [\alpha'_i] \cdots [\alpha'_n]0^m 0$$

by Lemma 61. The claim follows now by Lemma 63. $\square$

Definition 66 *Define* $\check{o}\colon \pi_0(\omega) \to \varepsilon_0$ *by*

$$\check{o}(0\alpha_1 0 \cdots 0\alpha_n 0^m 0) := \omega^{\check{o}(\alpha'_1)} + \cdots + \omega^{\check{o}(\alpha'_n)} + m,$$

where α'_i *is defined as in Lemma* 64.

Theorem 67 $\check{o}\colon \pi_0(\omega) \to \varepsilon_0$ *is an order-preserving isomorphism.*

PROOF. Define $\check{g} : \varepsilon_0 \to \pi_0(\omega)$ by

$$\check{g}(\omega^{\alpha_1} + \cdots + \omega^{\alpha_n} + m) := 0\check{g}(\alpha_1)''0 \cdots 0\check{g}(\alpha_n)''0^m 0,$$

where $\alpha_1 \geq \cdots \geq \alpha_n > 0$ and

$$\beta'' := \begin{cases} \langle j+1 \rangle \gamma^+ & \text{if } \beta = 0j\gamma 0 \text{ and } j \geq 1, \\ 1^k & \text{if } \beta = 0^k 0 \text{ for some } k. \end{cases}$$

Then we obviously have $\check{g} \circ \check{o} = \check{o} \circ \check{g} = id$. Note only that $(\alpha'')' = \alpha$ for every $\alpha \in \pi_0(\omega)$ and $(\beta')'' = \beta$ for every $\beta \in W_1$. That $\check{o}$ and $\check{g}$ are order-preserving follows from Lemma 65. $\square$

Remark 68 *It is somewhat interesting in the sense that the theorem above together with the definition of o gives simple and canonical order-preserving isomorphisms among* $\pi_0(\omega)$, $NF(W)$, *and* $NF \subseteq S$. *Indeed,* $\iota_1\colon \pi_0(\omega) \to NF(W)$ *and* $\iota_2\colon \pi_0(\omega) \to NF$ *are order-preserving isomorphisms:*

$$\iota_1(0\alpha_1 0 \cdots 0\alpha_n 0^m 0) := \alpha_1 0 \cdots 0\alpha_n 0^m$$

and

$$\iota_2(0\alpha_1 0 \cdots 0\alpha_n 0^m 0) := 0^m \alpha_n^* 0 \cdots 0\alpha_1^*,$$

where β^* *is the converse of* β.

4 A consequence of the structural equivalence

Having shown the structural equivalence, it is natural to expect that the behavior of each system in view of the slowly-well-orderedness is the same. As in case of the Cantor system we need some norm functions. Let $lh(\alpha)$ be the length of the word α and $ht(\alpha)$ a maximal component of α^+.

Definition 69 *$\check{N} : \pi_0(\omega) \to \mathbb{N}$ and $\hat{N} : NF(W) \to \mathbb{N}$ are defined as follows:*

$$\begin{aligned}\check{N}(0\alpha_1 0 \cdots 0\alpha_n 0^m 0) &:= \hat{N}(\alpha_1 0 \cdots 0\alpha_n 0^m) \\ &:= m + n \dot{-} 1 + \sum_{i=1}^{n} lh(\alpha_i) + \sum_{i=1}^{n}\sum_{k=0}^{n_i} a_{ik}\,,\end{aligned}$$

where $\alpha_i = a_{i0} \cdots a_{in_i} \in W_1$.

Roughly speaking, $\check{N}\alpha$ and $\hat{N}\alpha$ are the addition of the length of α and all of its components. Given $X \in \{\varepsilon_0, NF, NF(W), \pi_0(\omega)\}$ and $f\colon \mathbb{N} \to \mathbb{N}$ recall that $\mathrm{SWO}(X, \sqsubseteq_X, f)$ is defined as follows:

> for any k there exists a constant n which is so large that, for any finite sequence $\alpha_0, \ldots, \alpha_n$ from X with $\tilde{N}\alpha_i \le k + f(i)$ for all $i \le n$, there exist indices $\ell < m \le n$ satisfying $\alpha_\ell \sqsubseteq_X \alpha_m$.

Here $\tilde{N} \in \{N, \check{N}, \hat{N}\}$ and $\sqsubseteq_X \in \{\le, \le_0, \preceq\}$ depending on X.

Lemma 70 *(1) Let $\alpha \in \pi_0(\omega)$. Then*

$$N(\check{o}(\alpha)) \le \check{N}\alpha \quad \text{and} \quad \check{N}(\alpha^{+p}) \le (ht(\alpha) + p) \cdot N(\check{o}(\alpha)).$$

(2) Let $\alpha \in NF \cup NF(W)$. Then

$$N(o(\alpha)) \le \hat{N}\alpha \quad \text{and} \quad \hat{N}(\alpha^{+p}) \le (ht(\alpha) + p) \cdot N(o(\alpha)).$$

PROOF. It suffices to show (2). We write just N for $\hat{N}$ without causing no confusions. Let $\alpha = 0^m\alpha_1 0 \cdots 0\alpha_n \in NF$. We show the claim by induction on the maximal component in α. Note that $o(\alpha) = \omega^{o(\alpha_n^-)} + \cdots + \omega^{o(\alpha_1^-)} + m$.

If $n = 0$ it is obvious. Now assume $n > 0$.

$$\begin{aligned}N(o(\alpha)) &= m + n + \sum_{i=1}^{n} N(o(\alpha_i^-)) \\ &\le m + n + \sum_{i=1}^{n} N\alpha_i^- \quad \text{(by I.H.)} \le m + n - 1 + \sum_{i=1}^{n} N\alpha_i = N\alpha\end{aligned}$$

Further, we have

$$\begin{aligned}N\alpha^{+p} &= N(\alpha_1^{+p}) + \cdots + N(\alpha_n^{+p}) + p(m + n - 1) + (m + n - 1) \\ &= N((\alpha_1^-)^{+(p+1)}) + \cdots + N((\alpha_n^-)^{+(p+1)}) + (p + 1)(m + n - 1) \\ &\le (ht(\alpha_1^-) + p + 1) \cdot N(o(\alpha_1^-)) + \cdots + (ht(\alpha_n^-) + p + 1) \cdot N(o(\alpha_n^-)) \\ &\quad + (p + 1)(m + n - 1) \quad \text{(by I.H.)} \\ &\le (ht(\alpha) + p)(N(o(\alpha_1^-)) + \cdots + N(o(\alpha_n^-)) + m + n - 1) \\ &= (ht(\alpha) + p) \cdot N(o(\alpha))\end{aligned}$$

This completes the proof. □

This implies that the norm condition does not cause any essential difference in transformations between any two systems from ε_0, NF, $NF(W)$, or $\pi_0(\omega)$. Hence the following theorem is a direct consequence of Theorem 7 and Theorem 8.

Theorem 71 *Let* $\alpha \leq \varepsilon_0$.

(1) $\mathrm{SWO}(X, \sqsubseteq_X, f)$ *is* PRA-*provable for* $f(i) := |i| \cdot \mathrm{inv}(i)$.
(2) $\mathrm{SWO}(X, \sqsubseteq_X, g_n)$ *is not* PA-*provable for* $g_n(i) := |i| \cdot |i|_n$.
(3) Let $f_\alpha := |i| \cdot |i|_{H_\alpha^{-1}(i)}$. *Then* $\mathrm{SWO}(X, \sqsubseteq_X, f_\alpha)$ *is* PA-*provable iff* $\alpha < \varepsilon_0$.

Again let X be one of the systems ε_0, NF, $NF(W)$, $\pi_0(\omega)$. Given a function $f\colon \mathbb{N} \to \mathbb{N}$ let $\mathbf{EWD}(X, f)$ be defined as the Hydra game:

$$\mathbf{EWD}(X, f) :\equiv \forall \alpha \in X\, \exists n\, (\alpha(f, n) = \emptyset).$$

Theorem 72 *Let* $\alpha \leq \varepsilon_0$.

(1) $\mathbf{EWD}(X, \mathrm{inv})$ *is* PA-*provable.*
(2) $\mathbf{EWD}(X, g_n)$ *is not* PA-*provable for* $g_n(i) := |\cdot|_n$, $n \in \mathbb{N}$.
(3) Let $f_\alpha(i) := |\cdot|_{H_\alpha^{-1}(\cdot)}$. *Then* $\mathbf{EWD}(X, f_\alpha)$ *is* PA-*provable iff* $\alpha \in \varepsilon_0$.

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