Relationship between Kanamori-McAloon principle and Paris-Harrington theorem

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Abstract. We give a combinatorial proof of a tight relationship between the Kanamori-McAloon principle and the Paris-Harrington theorem with a number-theoretic parameter function. We show that the provability of the parametrised version of the Kanamori-McAloon principle can exactly correspond to the relationship between Peano Arithmetic and the ordinal ε_0 which stands for the proof-theoretic strength of Peano Arithmetic. Because A. Weiermann already noticed the same behaviour of the parametrised version of Paris-Harrington theorem, this indicates that both propositions behave in the same way with respect to the provability in Peano Arithmetic.

Keywords: Kanamori-McAloon principle, Paris-Harrington theorem, Peano Arithmetic, independence

1 Introduction

In combinatorics, the finite Ramsey theorem [1] shows that unavoidable structure exists in colourings of finite hypergraphs. In any colouring of the edges of a sufficiently large complete graph, we will always find monochromatic complete subgraphs. Below is a formal definition.

Given $X \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, let $[X]^n$ denote the set of all subsets of X with n elements. Each $m \in \mathbb{N}$ is identified with the set of its predecessors $\{0, \ldots, m-1\}$. If C is a colouring defined on $[X]^n$, we write $C(x_1, \ldots, x_n)$ for $C(\{x_1, \ldots, x_n\})$ assuming $x_1 < \cdots < x_n$. A set $H \subseteq X$ is called *C-homogeneous* if C is constant on $[H]^n$. We write

$$X \to (k)^n_c$$

if for all $C : [X]^n \to c$ there exists C-homogeneous $H \subseteq X$ s.t. $card(H) \geq k$. Let $R(n, c, k)$ be the least natural number ℓ such that $\ell \to (k)^n_c$.

The Paris-Harrington theorem [2] adds the requirement that H be *relatively large*, i.e., $card(H) \geq \min(H)$. If for all $C : [X]^n \to c$ there exists a relatively large C-homogeneous $H \subseteq X$ s.t. $card(H) \geq k$, we denote this fact by

$$X \to^* (k)^n_c. \tag{1}$$

Then put PH := $(\forall\, n, c, k)(\exists\, \ell)[\ell \rightarrow^* (k)^n_c]$. The independence of PH from Peano Arithmetic is obviously caused by the *relative largeness* condition because the finite Ramsey theorem itself is provable in a very weak system. Indeed, Erdős and Rado [3] showed the totality of R in $I\Delta_0 + (\exp)$:

Theorem 1 (Erdős and Rado [3]). *(In* $I\Delta_0 + (\exp)$*) Let* $n, c, k > 0$ *and* $k \geq n$*. Then*

$$R(n, c, k) \leq c * (c^{n-1}) * \cdots * (c^2) * (c(k-n)+1)\,.$$

Here $a * b := a^b$*, and* $*$ *is right-associative.*

That is, the upper bounds grows super-exponentially depending on the dimension n. Note that the right-hand side is approximately a tower of c's of height n.

Another Ramsey style proposition is introduced by Kanamori and McAloon [4]. A function $C : [X]^n \rightarrow \mathbb{N}$ is called *regressive* if $C(\bar{x}) < \min(\bar{x})$ for all $\bar{x} \in [X]^n$ such that $\min(\bar{x}) > 0$. A set $H \subseteq X$ is said to be *min-homogeneous* for C if, for all $s, t \in [H]^n$, $C(s) = C(t)$ holds whenever $\min(s) = \min(t)$. The fact that, given a regressive function C, such a min-homogeneous set H with $card(H) \geq k$ always exists is denoted by

$$X \rightarrow (k)^n_{reg}. \tag{2}$$

Put KM := $(\forall\, n, k)(\exists\, \ell)[\ell \rightarrow (k)^n_{reg}]$. KM is proved to be equivalent to PH, hence independent from Peano Arithmetic as well. Moreover, if we let PH^n and KM^n be the two propositions with fixed dimension n, then they are also equivalent. The following theorem summarises the main results in Paris and Harrington [2] and Kanamori and McAloon [4].

Theorem 2 ([2,4]). *Let* $n > 0$*.*

1. *(In* $I\Sigma_1$*)* PH, KM*, and the* 1*-consistency of* PA *are equivalent.*
2. *(In* $I\Sigma_1$*)* PH^{n+1}, KM^{n+1}*, and the* 1*-consistency of* $I\Sigma_n$ *are equivalent.*
3. PH^n *and* KM^n *are provable in* $I\Sigma_n$*.*
4. PH^{n+1} *and* KM^{n+1} *are* $I\Sigma_n$*-independent.*

The 1-consistency of a theory T is the assertion saying every T-provable Σ^0_1 sentence is true. We remark that the proofs in [2,4] are done model-theoretically.

A parametrised version of PH based on the growth rate of a number-theoretic function $f : \mathbb{N} \rightarrow \mathbb{N}$ is discussed in McAloon [5]. It results from replacing relative largeness with f-largeness, i.e. $card(H) \geq f(\min(H))$. We write $X \rightarrow^*_f (k)^n_c$ instead of (1) when the f-largeness holds. PH_f is then defined by

$$\mathrm{PH}_f := (\forall\, n, c, k)(\exists\, \ell)[\ell \rightarrow^*_f (k)^n_c].$$

PH^n_f is defined with fixed dimension n. Let R^*_f denote the Skolem function associated with PH_f: $R^*_f(n, c, k)$ is the least ℓ such that $\ell \rightarrow^*_f (k)^n_c$.

A parametrised version of KM based on the growth rate of a function is given in Kanamori and McAloon [4]. A function $C : [X]^n \rightarrow \mathbb{N}$ is called *f-regressive* if for all $\bar{x} \in [X]^n$ such that $f(\min(\bar{x})) > 0$ we have $C(\bar{x}) < f(\min(\bar{x}))$. We write $X \to (k)^n_{f\text{-}reg}$ instead of (2) to denote that, given a f-regressive function C, there exists always such a min-homogeneous set H for C with $card(H) \geq k$. KM_f is then denoted by

$$\mathrm{KM}_f :\equiv (\forall\, n, k)(\exists\, \ell)[\ell \to (k)^n_{f\text{-}reg}].$$

KM^n_f is defined with fixed dimension n. Let $R(\mu)_f$ denote the Skolem-function associated with KM_f: $R(\mu)_f(n,k)$ is the least ℓ such that $\ell \to (k)^n_{f\text{-}reg}$.

PH_f and KM_f are true for any function $f : \mathbb{N} \to \mathbb{N}$. They easily follow from König's lemma and the infinite version of Ramsey's theorem, cf. [4]. Note that both turn into the finite Ramsey theorem when f is a constant function. This and the PA-independence of PH and KM indicate that their PA-provability might depend on the growth rate of f. Indeed, Weiermann [6] showed that the PA-provability of PH_f depends on the growth rate of f by giving an interesting categorisation using Schwichtenberg-Buchholz-Wainer's fast-growing hierarchy [7,8,9].

In this paper, we introduce a way to establish the same result with respect to KM_f by revealing some relationship between KM_f and PH_f. Our results are based on some known results from [3,4] and an unpublished one by J. Paris. Let us first finish with preliminaries before abusing more undefined notions.

Peano Arithmetic (PA), is the first-order theory in the language with constants $0, 1, +, \cdot, <$, axioms defining the properties of these primitive notions, and the induction scheme for all formulae. A function $f : \mathbb{N}^d \to \mathbb{N}$ is provably recursive in PA if $f(\bar{m}) = n$ just holds when $\mathrm{PA} \vdash F(\bar{m}, n)$ for some formula $F(\bar{x}, y)$ which is Δ_1 in PA satisfying $\mathrm{PA} \vdash \forall\, \bar{x}\, \exists\, y\, F(\bar{x}, y)$. Here $\bar{x}$ stands for $x_1, ..., x_d$. $I\Sigma_n$ (resp. $I\Delta_0$) stands for Peano Arithmetic with the induction scheme for Σ_n (resp. Δ_0) formulae. (exp) denotes the totality of the exponential function: $\forall\, x\, \exists\, y\, (2^x = y)$.

2 The fast-growing hierarchy and the main theorems

We start with recalling Schwichtenberg-Buchholz-Wainer's fast-growing Hierarchy $(F_\alpha)_{\alpha \leq \varepsilon_0}$. Given a function $f : \mathbb{N} \to \mathbb{N}$ and natural number d, f^d denotes the d-th iteration of f, i.e. $f^0(x) := x$ and $f^{d+1}(x) := f(f^d(x))$. Then the fast-growing hierarchy is defined as follows:

$$F_0(x) := x + 1, \quad F_{\alpha+1}(x) := F^{x+1}_\alpha(x), \quad \text{and} \quad F_\lambda(x) := F_{\lambda[x]}(x)$$

Here $\cdot\,[\cdot] : \varepsilon_0 \times \mathbb{N} \to \varepsilon_0$ is a fixed assignment of fundamental sequences to ordinals below ε_0 based on the Cantor Normal Form. $(\gamma + \omega^\lambda)[x] := \gamma + \omega^{\lambda[x]}$, $(\gamma + \omega^{\beta+1})[x] := \gamma + \omega^\beta \cdot x$, where $\alpha_0(x) := x$, $\alpha_{d+1}(x) := \alpha^{\alpha_d(x)}$ and $\alpha_d := \alpha_d(1)$. For technical reasons, we put $\varepsilon_0[x] := \omega_x$, $(\beta + 1)[x] := \beta$ and $0[x] := 0$.

Note that $F_1(x) = 2x$, $F_2(x) = 2^x \cdot x$, and $F_3(x) \geq 2_x(x)$. The function hierarchy $(F_\alpha)_{\alpha < \varepsilon_0}$ plays an important role in proof theory as the following theorem shows the close relationship to PA, which is a folklore in proof theory.

We say a function f *captures* or *eventually dominates* another function g if there is an m such that $g(i) \leq f(i)$ for any $i \geq m$.

Theorem 3 ([7,8,9]). *Let* $d > 0$. *Then*

1. $\mathrm{PA} \vdash \forall x \exists y [F_\alpha(x) = y]$ *iff* $\alpha < \varepsilon_0$.
2. *Let* f *be a* Σ_1*-definable function. Then* PA *proves the totality of* f *if and only if* f *is primitive recursive in* F_α *for some* $\alpha < \varepsilon_0$.
3. F_{ε_0} *eventually dominates all* PA*-provably recursive functions.*

Based on Theorem 3, Weiermann [6] gave a classification of the provability threshold of PH using the following parameter functions:

$$|x| = \lfloor \log_2(x+1) \rfloor, \quad |x|_{d+1} = ||x|_d|, \quad \text{and} \quad \log^* x := \min\{d \mid |x|_d \leq 2\}.$$

For convenience, we put $|x|_0 = x$ and $\log_2 0 := 0$. For an unbounded function $f : \mathbb{N} \longrightarrow \mathbb{N}$ we denote by f^{-1} the inverse of f, i.e. $f^{-1}(x) := \min\{y : f(y) > x\}$. Note that $f^{-1}(x) \leq y$ iff $x < f(y)$. Given $\alpha \leq \varepsilon_0$, define

$$f_\alpha(i) := |i|_{F_\alpha^{-1}(i)}\,.$$

Then the PA-provability of PH_{f_α} corresponds exactly to the relationship between PA and the ordinal ε_0.

Theorem 4 (Weiermann [6]).

1. $R^*_{\log^*}$ *is provably total in* $I\Sigma_1$*, hence* $I\Delta_0 + (\exp) \vdash \mathrm{PH}_{\log^*}$.
2. *For all* $d \in \mathbb{N}$*, the totality of* $R^*_{|\cdot|_d}$ *is not provable in* PA*, hence* $\mathrm{PH}_{|\cdot|_d}$ *is* PA*-independent.*
3. *The totality of* $R^*_{f_\alpha}$ *is provable in* PA *iff* $\alpha < \varepsilon_0$*, hence* PH_{f_α} *is* PA*-provable iff* $\alpha < \varepsilon_0$.

Proof. The provability part follows easily from Erdős and Rado's upper bound for the standard Ramsey theorem in Theorem 1. For the unprovability part, Weiermann constructed some interesting colouring functions which are refinements of the ordinal-based colourings by Loebl and Nešetřil [10]. In fact, he showed

- $R^*_{|\cdot|_{n-2}}(n+1, 3_n(n+k+3), k^{(n)}) \geq F_{\omega_{n-1}(k)}(k-1)$ for any $n \geq 2$, $k \geq 4$,
- $R^*_{f_{\varepsilon_0}}(n+1, 3_n(2n+3), n^{(n)}) \geq F_{\varepsilon_0}(n-2)$ for any $n \geq 4$,

where $k^{(i)} := k + 3 + 3^2 + \cdots + 3^i$. □

Remark 5. Characterisation of the PA-provability of some propositions using parameter functions in the form of f_α is first introduced in Arai [11].

Corollary 6 (Weiermann [6]). *There exist two primitive recursive functions* $p, q : \mathbb{N} \to \mathbb{N}$ *such that*

$$R^*_{id}(n, p(n), q(n)) \geq F_{\varepsilon_0}(n-3)$$

for any $n \geq 5$.

Our main theorems say that the same classification of parameter functions for KM_f holds:

1. $R(\mu)_{\log^*}$ is provably total in $I\Sigma_1$, hence $I\Delta_0 + (\exp) \vdash \mathrm{KM}_{\log^*}$.
2. For all $d \in \mathbb{N}$, the totality of $R(\mu)_{|\cdot|_d}$ is not provable in PA, hence $\mathrm{KM}_{|\cdot|_d}$ is PA-independent.
3. The totality of $R(\mu)_{f_\alpha}$ is provable in PA iff $\alpha < \varepsilon_0$, hence KM_{f_α} is PA-provable iff $\alpha < \varepsilon_0$.

The proofs will be divided into Theorem 8, Theorem 12, and Theorem 13.

3 Provability

We start with a simple proof of the provability of $\mathrm{KM}_{\log^*}$ and KM_{f_α} with $\alpha < \varepsilon_0$.

Lemma 7. *Given an unbounded function $f : \mathbb{N} \to \mathbb{N}$, set $f' := |\cdot|_{f^{-1}(\cdot)}$. Then there is a primitive recursive function $p : \mathbb{N}^2 \to \mathbb{N}$ satisfying*

$$R_{f'}(n,k) \leq 2_n(f(p(n,k))) + f(p(n,k))$$

for all $n, k > 0$.

Proof. We may assume w.l.o.g. that $f(x) \geq x$ for any $x \in \mathbb{N}$. Given n, $k > 0$, assume $n \leq k$ and let $p(n,k)$ be the least $x > n$ such that

$$L(n,k) := (c) * (c^{n-1}) * \cdots * (c^2) * (c(k-n)+1) < 2_n(c)\,,$$

where $c := f(x)$. The existence of such an x can be seen by noting that the right-hand side is approximately a tower of c of height $n+1$ while the left-hand side is approximately of height n. Indeed we can show for $c \geq 2^{2(n+k)}$ that $L(n,k) \leq c_n(n) \leq 2_n(c)$. That is, $p(n,k) := 4^{n+k}$ is a good candidate.

Put $m := 2_n(f(p(n,k))) + f(p(n,k))$. Then $m \leq 2_{n+1}(f(p(n,k)))$. Indeed, $2_n(i) + i \leq 2_{n+1}(i)$ for all n, i. Let $C : [m]^n \to \mathbb{N}$ be any f'-regressive function. Define a new f'-regressive function

$$D : [f(p(n,k)), m]^n \to \mathbb{N}$$

by restricting C. Note that, for any $y \in [f(p(n,k)), m]$, it holds

$$f'(y) \leq |2_{n+1}(f(p(n,k)))|_{f^{-1}(f(p(n,k)))} = |2_{n+1}(f(p(n,k)))|_{p(n,k)+1} < f(p(n,k))\,.$$

This implies that $\mathrm{Im}(D) \subseteq f(p(n,k))$. By Theorem 1, there is a D-homogeneous set Y, hence min-homogeneous for C such that $card(Y) \geq k$. ◻

It is now obvious to see the provability part. Note that $\log^*(n) \leq Exp_s^{-1}(n)$, where Exp_s denotes the super-exponential function defined by $Exp_s(n) := 2_n(1)$.

Theorem 8. *Let $\alpha < \varepsilon_0$.*

1. $\mathrm{KM}_{\log^*}$ *is provable in* $I\Delta_0 + (\exp)$.
2. KM_{f_α} *is provable in* PA.

Proof. Both claims follow from Lemma 7 together with Theorem 3. ◻

4 Independence

We now turn our attention to the independent proof. We start with recalling a lemma.

Lemma 9 (Kanamori and McAloon [4]). *Let $I \subseteq \mathbb{N}$. If $C : [I]^n \to \mathbb{N}$ is regressive, then $H \subseteq I$ is min-homogeneous for C iff every $Y \subseteq H$ of cardinality $n+1$ is min-homogeneous for C.*

The following lemma is slightly modified from Lemma 3.3 in [4]. Given a function $f : \mathbb{N} \to \mathbb{N}$, the function $|f|$ is defined by $|f|(i) = |f(i)|$. Given $x \in \mathbb{N}$, note that any $y < x$ can be represented as $(y_0, \ldots, y_{d-1})_2$ in binary notation, where $d = |x|$. Put $C(\bar{x}) = (C_0(\bar{x}), \ldots, C_{d-1}(\bar{x}))_2$, where $\bar{x} \in [x, y]^n$ and $d = |f(\min(\bar{x}))|$.

Lemma 10. *If $C : [\ell]^n \to y$ is f-regressive and $f(i) \le i$, then there is a regressive function C' such that*

- *$C' : [\ell]^{n+1} \to y$ is $(2|f|+1)$-regressive, and*
- *if H' is min-homogeneous for C', then $H = H' - (7 \cup \{\max(H')\})$ is min-homogeneous for C.*

Proof. Define C' on $[x, y]^{n+1}$ as follows:

- $C'(x_0, \ldots, x_n) = 0$ if either $x_0 < 7$, or $\{x_0, \ldots, x_n\}$ is min-homogeneous for C;
- $C'(x_0, \ldots, x_n) = 2i + C_i(x_0, \ldots, x_{n-1}) + 1$, otherwise, where $i < |f(x_0)|$ is the least such that $\{x_0, \ldots, x_n\}$ is not min-homogeneous for C_i.

Then C' is $2|f|+1$-regressive and even regressive because $2|f(x_0)| + 1 \le 2|x_0| + 1 \le x_0$ for any $x_0 \ge 7$. Suppose H' is min-homogeneous for C' and H is as described. If $C' \restriction [H]^{n+1} = \{0\}$, then we are done by the previous lemma. Suppose on the contrary that there were $x_0 < \cdots < x_n$ all in H such that $C'(x_0, \ldots, x_n) = 2i + C_i(x_0, \ldots, x_{n-1}) + 1$. Given any $\bar{s}$, $\bar{t} \in [\{x_0, \ldots, x_n\}]^n$ with $\min(\bar{s}) = \min(\bar{t}) = x_0$, note that

$$C'(\bar{s} \cup \{\max(H')\}) = C'(x_0, \ldots, x_n) = C'(\bar{t} \cup \{\max(H')\})$$

by the min-homogeneity of H'. But then, $C_i(\bar{s}) = C_i(\bar{t})$, so that $\{x_0, \ldots, x_n\}$ were min-homogeneous for C_i after all, which contradicts the assumption. □

Now we are going to apply Lemma 10 iteratively. Define a sequence of functions as follows:

$$g_0(i) := i \quad \text{and} \quad g_{m+1}(i) := 2 \cdot |g_m(i)| + 1$$

Lemma 11. *Let n, $m \ge 1$. Then, in $I\Sigma_1$, $\mathrm{KM}^{n+m}_{g_m}$ implies KM^n.*

Proof. Assume $\mathrm{KM}_{g_m}^{n+m}$. Let k be given, then by assumption we can find ℓ satisfying
$$\ell \to (k+m+7)^{n+m}_{g_m\text{-}reg}\,.$$
We claim $\ell \to (k)^n_{reg}$ holds. Suppose $C : [\ell]^n \to \ell$ is regressive. By applying the previous lemma m times, we get a g_m-regressive function $C' : [\ell]^{n+m} \to \ell$. Then there is a set $H' \subseteq \ell$, min-homogeneous for C', such that $card(H) \geq k+m+7$, so
$$H = H' - 7 \cup \{\text{the last } m \text{ elements of } H'\}$$
is min-homogeneous for C and $card(H) \geq k$. □

Theorem 12. *Let d be a natural number.*

1. *In $I\Sigma_1$, $\mathrm{KM}_{|\cdot|_d}$ implies* KM.
2. $\mathrm{KM}_{|\cdot|_d}$ *is* PA*-independent, i.e. $R(\mu)_{|\cdot|_d}$ is not provably total in* PA.

Proof. Note just that $\mathrm{KM}_{g_{d+1}}$ follows from $\mathrm{KM}_{|\cdot|_d}$. Therefore, $R(\mu)_{|\cdot|_d}$ cannot be captured by F_{ε_0}. □

It remains to prove the PA-independence of $\mathrm{KM}_{f_{\varepsilon_0}}$. Here we introduce two ways to prove it. The first one is based on an unpublished, combinatorial proof[1] by J. Paris of $(\mathrm{KM}^n \to \mathrm{PH}^n)$ by showing the following:

$$I\Delta_0 + (\exp) \vdash (\mathrm{KM}^n \to \mathrm{KM}^n_{Exp_2}) \ \wedge \ (\mathrm{KM}^n_{Exp_2} \to \mathrm{PH}^n)$$

for any $n \geq 2$, where $Exp_2(x) := 2^{2^x}$.

The second one is based on a refinement of Paris' proof. Indeed, we will give a direct, combinatorial proof of $(\mathrm{KM}^n \to \mathrm{PH}^n)$. The proof is not just a composition of two proofs given by Paris, but a real refinement of it.

The point here is that both proofs given by Paris and us are purely combinatorial and hence provide us with a primitive recursive function $p_1 : \mathbb{N}^3 \to \mathbb{N}$ satisfying
$$R(\mu)(n, p_1(n,c,k)) \geq R^*(n,c,k) \tag{3}$$
for any $n \geq 2$ and for any $c, k \geq 0$. And we can use this fact to prove the PA-independence of KM_{f_α}. We first show how to make use of (3) and then present a sketch of our refined proof in Appendix.

Theorem 13. $\mathrm{KM}_{f_{\varepsilon_0}}$ *is* PA*-independent.*

Proof. From the proof of Lemma 11, we know

$$R(\mu)_{|\cdot|_d}(n+d+1, k+d+8) \geq R(\mu)_{g_{d+1}}(n+d+1, k+d+8) \geq R(\mu)(n,k)$$

for any positive n, d, k. Using this and (3), it follows that

$$R(\mu)_{|\cdot|_d}(n+d+1, p_1(n,c,k)+d+8) \geq R^*(n,c,k),$$

[1] This is mentioned in Kanamori-McAloon [4], page 39, and we got a copy of the proof.

and hence

$$R(\mu)_{|\cdot|_d}(n+d+1, p_1(n, p(n), q(n)) + d + 8) \geq F_{\varepsilon_0}(n-3)$$

for any $n \geq 5$, where p, q are some primitive recursive functions satisfying Corollary 6. This indicates the existence of a primitive recursive function $p' : \mathbb{N} \to \mathbb{N}$ such that, putting $d := n$,

$$R(\mu)_{|\cdot|_n}(2n+1, p'(n)) \geq F_{\varepsilon_0}(n-3) \tag{4}$$

for any $n \geq 5$. We use this fact to show that PA cannot prove the totality of $R(\mu)_{f_{\varepsilon_0}}$. Let $n \geq 5$ and p' be a primitive recursive function satisfying (4). We claim

$$R(\mu)_{f_{\varepsilon_0}}(2n+1, p'(n)) > F_{\varepsilon_0}(n-3),$$

which implies that $R_{f_{\varepsilon_0}}$ cannot be provably recursive in PA.

Assume otherwise. Note first that by (4), there is some $|\cdot|_n$-regressive function $G : [F_{\varepsilon_0}(n-3) - 1]^{2n+1} \to \mathbb{N}$ which has no min-homogeneous set of cardinality $p'(n)$. On the other hand, G is f_{ε_0}-regressive. In fact, for all $i < F_{\varepsilon_0}(n-3)$ it holds that $F_{\varepsilon_0}^{-1}(i) \leq n-3$, hence $|i|_n \leq |i|_{n-3} \leq |i|_{F_{\varepsilon_0}^{-1}(i)}$. This means G should have a min-homogeneous set of cardinality $p'(n)$, which is not possible. □

Finally, Theorem 8, Theorem 12, and Theorem 13 establish that PH_f and KM_f behave in the same manner with respect to the provability in PA.

Theorem 14. *Let $f_\alpha(i) := |i|_{F_\alpha^{-1}(i)}$, where $\alpha \leq \varepsilon_0$. Then*

1. *Both $\mathrm{KM}_{\log^*}$ and $\mathrm{PH}_{\log^*}$ are PA-provable.*
2. *For all $d \in \mathbb{N}$, both $\mathrm{KM}_{|\cdot|_d}$ and $\mathrm{PH}_{|\cdot|_d}$ are PA-independent.*
3. *KM_{f_α} (resp. PH_{f_α}) is PA-provable if and only if $\alpha < \varepsilon_0$.*

5 Conclusion

What we have shown in this paper is about the *global* relationship between two propositions KM_f and PH_f with respect to the provability in PA. Contrary to the same behaviour in the global level, they behave differently in the local level as shown by Carlucci, Lee, and Weiermann [12] where it is shown that

$$I\Sigma_n \vdash \mathrm{KM}^{n+1}_{|\cdot|_d} \text{ if and only if } d \geq n,$$

while Weiermann [13] showed that

$$I\Sigma_n \vdash \mathrm{PH}^{n+1}_{|\cdot|_d} \text{ if and only if } d \geq n+1.$$

In Kojman, Lee, Omri, and Weiermann [14] one can see some refinements of the results about Kanamori-McAloon principle in case $n = 1$.

The methodology used in current paper is different from that of [12] although the results in the latter are refinements of those in this paper. Here we investigated the global relationship between PH and KM in a direct way, while the results in [12] and [13] are compared indirectly. We also refer to Bovykin [15] where related results on the Kanamori-McAloon principle are introduced using a model-theoretic approach *à la* Paris and Harrington [2].

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Appendix

Here we give a combinatorial proof of KM^n implies PH^n. The basic idea is getting min-homogeneous sets of very large cardinality such that some fine thinning can be chosen whose every two elements lie sufficiently far away from each other. This is also one of the basic ideas of Paris' original proof. We shall demand somewhat more, and this will be achieved by using the following lemma for the construction of such sets. The following lemma from [4] is crucial. We present the original proof again because the proof itself plays an important role later.

Lemma 15 (Kanamori and McAloon [4]). *There are three regressive functions $\eta_1, \eta_2, \eta_3 : [\mathbb{N}]^2 \to \mathbb{N}$ such that whenever H' is min-homogeneous for all of them, then*

$$H = H' \setminus \{\textit{the last three elements of } H'\}$$

has the property that $x < y$ both in H implies $x^x \leq y$.

Proof. Define $\eta_1, \eta_2, \eta_3 : [\mathbb{N}]^2 \to \mathbb{N}$ by:

$$\eta_1(x,y) = \begin{cases} 0 & \text{if } 2x \leq y\,, \\ y \dot{-} x & \text{otherwise,} \end{cases}$$

$$\eta_2(x,y) = \begin{cases} 0 & \text{if } x^2 \leq y\,, \\ u & \text{otherwise, where } u \cdot x \leq y < (u+1) \cdot x\,, \end{cases}$$

$$\eta_3(x,y) = \begin{cases} 0 & \text{if } x^x \leq y\,, \\ v & \text{otherwise, where } x^v \leq y < x^{v+1}\,. \end{cases}$$

Suppose that H' is as hypothesised, and let $z_1 < z_2 < z_3$ be the last three elements of H'. If $x < y$ are both in $H' \setminus \{z_3\}$, then since $\eta_1(x,y) = \eta_1(x,z_3)$, clearly we must have $\eta_1(x,y) = 0$. Hence, η_1 on $[H' \setminus \{z_3\}]^2$ is constantly 0.

Next, assume that $x < y$ are both in $H' \setminus \{z_2, z_3\}$ and $\eta_2(x,y) = u > 0$. then $u \cdot x \leq y < z_2 < u \cdot x + x$ by min-homogeneity, and so we have $u \cdot x + x \leq y + x \leq y + y \leq z_2$ by the previous paragraph. But this leads to the contradiction $z_2 < z_2$. Hence, η_2 on $[H' \setminus \{z_2, z_3\}]^2$ is constantly 0.

Finally, we can iterate the argument to show that η_3 on $[H' \setminus \{z_1, z_2, z_3\}]^2$ is constantly 0, and so the proof is complete. □

Theorem 16 (In $I\Sigma_1$). KM^n *implies* PH^n *for all* $n \geq 2$.

Proof. For a better readability we prove the claim for $n = 3$. This case is general enough, and a proof for arbitrary n needs so many symbols that it would be difficult to see the ground idea of the proof. A general proof is presented in the author's PhD thesis [16].

Assume KM^n. Given z, k, put $\ell := 21z + 22$. Then we can find m and y satisfying $m \to (\ell + 7)^3_4$ and $[x,y] \to (m)^3_{reg}$, where x is chosen such that $x \geq \max\{7, z, k, M\}$ and M is so large that any $i \geq M$ satisfies

$$2|i| + 1 < i, \quad 2^{2^{|i|_3}} < i, \quad \text{and} \quad |i|_3 > 2\,. \tag{5}$$

How large x should be will become obvious during the proof below. Notice that the existence of m is guaranteed by the finite Ramsey theorem, and y can be chosen such that $y \to (m+x)^3_{reg}$. We claim $y \to^* (k)^3_z$. Indeed, we are going to show $[x,y] \to^* (k)^3_z$.

Let a function $f : [x,y]^3 \to z$ be given. We will define a regressive function $g : [x,y]^3 \to y$ such that a fine thinning of a min-homogeneous set for g leads to a f-homogeneous set of cardinality k. We need some preparation. Let α, β, γ be numbers from $[x,y]$ satisfying $\alpha < \beta < \gamma$. Below we will consider finite sequences as functions with finite domains.

– First construct a finite sequence Q_γ of length at most $(\gamma - x)$ by letting $Q_\gamma(0) := x$, $Q_\gamma(1) := x+1$, and, assuming $Q_\gamma(i-1)$ is defined, $Q_\gamma(i)$ be the least t such that $Q_\gamma(i-1) < t < \gamma$ and

$$\forall j,\, p < i\, [\, j < p \to f(Q_\gamma(j), Q_\gamma(p), t) = f(Q_\gamma(j), Q_\gamma(p), \gamma)\,]\,.$$

If there is no such t, then the construction stops. Now put

$$R_{\gamma\alpha}(j,p) := f(Q_\gamma(j), Q_\gamma(p), \gamma)$$

for $j, p \in \alpha \cap \mathrm{dom}(Q_\gamma)$, with $j < p$. Notice that $Q_\gamma \restriction \alpha$ can be regained from f and $R_{\gamma\alpha}$, i.e., γ is not necessary.

– If $\beta \in \mathrm{dom}(Q_\gamma)$, construct another sequence $P_{\gamma\beta}$ by letting $P_{\gamma\beta}(0) := x$, and, assuming $P_{\gamma\beta}(i-1)$ is defined, $P_{\gamma\beta}(i) :=$ be the least t such that $P_{\gamma\beta}(i-1) < t$, $t \in \mathrm{Im}(Q_\gamma \restriction \beta)$, and

$$\forall j < i (f(P_{\gamma\beta}(j), t, \gamma) = f(P_{\gamma\beta}(j), Q_\gamma(\beta), \gamma))\,.$$

If there is no such t, then the construction stops. Now put

$$S_{\gamma\beta\alpha}(j) := f(P_{\gamma\beta}(j), Q_\gamma(\beta), \gamma)$$

for $j \in \alpha \cap \mathrm{dom}(P_{\gamma\beta})$. Notice that $P_{\gamma\beta} \restriction \alpha$ can be regained from f, $S_{\gamma\beta\alpha}$ and γ.

– Let η_j, $j = 1, 2, 3$, be the three regressive functions from Lemma 15. Applying Lemma 10, define $\bar{\eta}_j : [x,y]^3 \to 2|y|+1$, $j = 1, 2, 3$, such that, if $\bar{H}$ min-homogeneous for all $\bar{\eta}_j$, then $H := \bar{H} - \{\text{the last element of } \bar{H}\}$ is min-homogeneous for all η_j. (Note that $x \geq 7$.) Define a function h as follows:[2]

$$h(\alpha,\beta,\gamma) := \begin{cases} 0 & \text{if } \bar{\eta}_j(\alpha,\beta,\gamma) = 0 \text{ for each } j \in \{1,2,3\}, \\ j & \text{if } j \text{ is the least one s.t. } \bar{\eta}_j(\alpha,\beta,\gamma) \neq 0. \end{cases}$$

[2] Here is the place where a refinement of Paris' original proof takes place.

We can now define g on $[x,y]^3$:

$$g(\alpha,\beta,\gamma) := \begin{cases} \bar{\eta}_j(\alpha,\beta,\gamma) \text{ if } h(\alpha,\beta,\gamma)=j>0, \\ 0 \qquad\qquad\quad \text{otherwise and } \neg(x \le |\alpha|_3 < |\beta|_3 < |\gamma|_3), \\ \langle R_{|\gamma|_3|\alpha|_3},\ |\gamma|_3 \pmod{2z|\alpha|_3}\rangle_2 \\ \qquad\qquad\qquad \text{otherwise and } |\beta|_3 \notin \mathrm{dom}(Q_{|\gamma|_3}), \\ \langle R_{|\gamma|_3|\alpha|_3},\ S_{|\gamma|_3|\beta|_3|\alpha|_3},\ |\gamma|_3 \pmod{2z|\alpha|_3}\rangle_3 \\ \qquad\qquad\qquad \text{otherwise.} \end{cases}$$

Here we assume that R and S are coded as natural numbers by suitable encoding functions $\langle -,-\rangle_2$ and $\langle -,-,-\rangle_3$ satisfying the following:

$$\langle R_{\gamma\alpha}, \gamma \pmod{2z\alpha}\rangle_2,\ \langle R_{\gamma\alpha},\ S_{\gamma\beta\alpha},\ \gamma \pmod{2z\alpha}\rangle_3 \le 2^{2^\alpha} \tag{6}$$

for all $\alpha \ge x$. Notice that $\mathrm{dom}(R_{\gamma\alpha}) \subseteq \alpha\times\alpha$, $\mathrm{dom}(S_{\gamma\beta\alpha}) \subseteq \alpha$ and $\mathrm{Im}(R_{\gamma\alpha}) \cup \mathrm{Im}(S_{\gamma\beta\alpha}) \subseteq z \le x$.

This is true if x is large enough. From now on we assume that (6) is always satisfied for any $\alpha \ge x$. This implies that g is regressive:

$$g(\alpha,\beta,\gamma) \le \max\{\bar{\eta}_j(\alpha,\beta,\gamma), 2^{2^{|\alpha|_3}}\} < \alpha\,.$$

Notice that $\alpha \ge x \ge M$.

Let X_0 be min-homogeneous for g and homogeneous for h such that $card(X_0) \ge \ell + 7$. Define X_1 and X by

$$X_1 := X_0 - \{\text{the last four elements of } X_0\}\,,$$
$$X := X_1 - \{\text{the first three elements of } X_1\}\,,$$

then $card(X) \ge \ell$. Let also Y' be the set of every third element of X and Y the set of every second element of Y', i.e. Y is the set of every 6th element of X, so $card(Y) \ge \ell/7 > 3z+3$. Now we show a series of claims.

Claim 1: $h \restriction [X_1]^3$ is the constant function with value 0.

Proof of Claim 1: Let $a<b<c<d$ be the last four elements of X_0 and assume $h \restriction [X_1]^3 = 1$. Then $h \restriction [X_0]^3 = 1$ and $g \restriction [X_0]^3 = \bar{\eta}_1 \restriction [X_0]^3$. It follows that X_0 is min-homogeneous for $\bar{\eta}_1$, so $X_0 \setminus \{d\}$ is min-homogeneous for η_1. By the proof of Lemma 15 $\eta_1 \restriction [X_0 \setminus \{c,d\}]^2 = 0$. Hence $\bar{\eta}_1 \restriction [X_0 \setminus \{c,d\}]^3 = 0$ contradicting $h \restriction [X_0]^3 = 1$. Therefore, $h \restriction [X_0]^3 \ne 1$ and $\bar{\eta}_1 \restriction [X_0]^3 = 0$ because it is a constant function. In particular, X_0 is min-homogeneous for $\bar{\eta}_1$, and so $\eta_1 \restriction [X_0 \setminus \{c,d\}]^2 = 0$.

Finally, we can iterate the same argument to show that $\eta_2 \restriction [X_0 \setminus \{b,c,d\}]^2 = 0$ and $\eta_3 \restriction [X_0 \setminus \{a,b,c,d\}]^2 = 0$. It follows that $h \restriction [X_1]^3 \notin \{1,2,3\}$, and so we should have $h \restriction [X_1]^3 = 0$. q.e.d.

Claim 2: $g \restriction [X]^3 > 0$.

Proof of Claim 2: By Lemma 10 and Lemma 15, for all $\alpha < \beta \in X_1$, $2^\alpha < \beta$, and hence $|\alpha|_3 < |\beta|_3$ if $|\alpha|_3 > 2$. Because there are three elements from X_1 which are smaller than $\min(X)$, we also have $x \leq |\alpha|_3$ for all $\alpha \in X$. q.e.d.

Claim 3: Let $\alpha < \beta < \delta < \gamma$ be from Y'. Then $z|\alpha|_3 < |\delta|_3 - |\beta|_3$.

Proof of Claim 3: Let $\tau < \rho \in X$ such that $\alpha < \tau < \rho < \beta < \delta$. Because $g(\alpha, \tau, \rho) = g(\alpha, \tau, \beta) = g(\alpha, \tau, \delta)$ by min-homogeneity of $X \subseteq X_0$ we have

$$|\rho|_3 \pmod{2z|\alpha|_3} = |\beta|_3 \pmod{2z|\alpha|_3} = |\delta|_3 \pmod{2z|\alpha|_3}.$$

Then for some $n_1 < n_2 \in \mathbb{N}$, $|\beta|_3 = |\rho|_3 + 2z|\alpha|_3 \cdot n_1$ and $|\delta|_3 = |\rho|_3 + 2z|\alpha|_3 \cdot n_2$. Therefore, $|\delta|_3 - |\beta|_3 > z|\alpha|_3$. q.e.d.

Claim 4: Let $\alpha < \beta < \delta < \gamma$ be from Y'. Then $|\beta|_3 \in \operatorname{dom}(Q_{|\gamma|_3})$, $Q_{|\gamma|_3}(|\beta|_3) < |\delta|_3$, $Q_{|\gamma|_3} \upharpoonright |\beta|_3 = Q_{|\delta|_3} \upharpoonright |\beta|_3$, and $\operatorname{dom}(R_{|\gamma|_3|\alpha|_3}) = |\alpha|_3$.

Proof of Claim 4: Let $\tau, \rho \in X$ such that $\beta < \tau < \rho < \delta$. Because $g(\beta, \tau, \rho) = g(\beta, \tau, \gamma)$ by min-homogeneity of $X \subseteq X_0$ we have $R_{|\rho|_3|\beta|_3} = R_{|\gamma|_3|\beta|_3}$ and $Q_{|\rho|_3} \upharpoonright |\beta|_3 = Q_{|\gamma|_3} \upharpoonright |\beta|_3$. Therefore, for each $j < p \in |\beta|_3 \cap \operatorname{dom}(Q_{|\rho|_3}) = |\beta|_3 \cap \operatorname{dom}(Q_{|\gamma|_3})$ it holds that

$$f(Q_{|\rho|_3}(j), Q_{|\rho|_3}(p), |\rho|_3) = f(Q_{|\gamma|_3}(j), Q_{|\gamma|_3}(p), |\gamma|_3).$$

Let $\mu := |\beta|_3 \cap \operatorname{dom}(Q_{|\rho|_3})$. Then $\mu = |\beta|_3$, since otherwise it would follow that $Q_{|\gamma|_3}(\mu)$ is defined. Note that $|\rho|_3 < |\gamma|_3$. This contradicts the definition of μ. Here we used the fact that $Q_{|\rho|_3} \upharpoonright \mu = Q_{|\gamma|_3} \upharpoonright \mu$. In the same way, we can show $|\beta|_3 \in \operatorname{dom}(Q_{|\gamma|_3})$ and $Q_{|\gamma|_3}(|\beta|_3) \leq |\rho|_3 < |\delta|_3$. Replacing $|\gamma|_3$ with $|\delta|_3$, we get $Q_{|\delta|_3} \upharpoonright |\beta|_3 = Q_{|\gamma|_3} \upharpoonright |\beta|_3$. This implies that $\operatorname{dom}(R_{|\gamma|_3|\alpha|_3}) = |\alpha|_3$. q.e.d.

Claim 5: Let $\alpha < \beta < \delta < \gamma < \eta$ be from Y. Then $\operatorname{dom}(S_{|\gamma|_3|\beta|_3|\alpha|_3}) = |\alpha|_3$ and $P_{|\gamma|_3|\beta|_3}(|\alpha|_3) < |\beta|_3$.

Proof of Claim 5: Let $\tau \in Y'$ be such that $\alpha < \tau < \beta$. Then $g(\alpha, \tau, \gamma) = g(\alpha, \beta, \gamma)$ by min-homogeneity of $Y \subseteq X_0$. By the same arguments as above, we can show $S_{|\gamma|_3|\tau|_3|\alpha|_3} = S_{|\gamma|_3|\beta|_3|\alpha|_3}$, i.e., $\mu := |\alpha|_3 \cap \operatorname{dom}(P_{|\gamma|_3|\tau|_3}) = |\alpha|_3 \cap \operatorname{dom}(P_{|\gamma|_3|\beta|_3})$ and for each j, u, with $j < \mu$,

$$f(P_{|\gamma|_3|\tau|_3}(j), Q_{|\gamma|_3}(|\tau|_3), |\gamma|_3) = f(P_{|\gamma|_3|\beta|_3}(j), Q_{|\gamma|_3}(|\beta|_3), |\gamma|_3).$$

As in the proof of Claim 4, we can show that $\mu = |\alpha|_3$ and $P_{|\gamma|_3|\beta|_3}(|\alpha|_3) \leq Q_{|\gamma|_3}(|\tau|_3) < |\beta|_3$. q.e.d.

Claim 6: Let $\alpha < \beta < \delta < \gamma < \eta$ be from Y. Then $P_{|\gamma|_3|\delta|_3}(|\alpha|_3) < |\beta|_3$ and $P_{|\delta|_3|\beta|_3} \upharpoonright |\alpha|_3 = P_{|\eta|_3|\gamma|_3} \upharpoonright |\alpha|_3$.

Proof of Claim 6: Let τ be as above. Replacing β with δ above, it holds that $P_{|\gamma|_3|\delta|_3}(|\alpha|_3) \leq Q_{|\gamma|_3}(|\tau|_3) < |\beta|_3$. Then $P_{|\delta|_3|\beta|_3} \upharpoonright |\alpha|_3 = P_{|\eta|_3|\gamma|_3} \upharpoonright |\alpha|_3$ follows directly from $g(\alpha, \beta, \delta) = g(\alpha, \gamma, \eta)$. q.e.d.

Let $\alpha_0, \alpha_1, \ldots, \alpha_p$ enumerate all the elements of Y in the ascending order. Then choose a set, for each $i < [p/3] - 1$,

$$Z_i \subseteq \operatorname{Im}(P_{|\alpha_p|_3|\alpha_{p-1}|_3} \upharpoonright [\,|\alpha_{3i}|_3, |\alpha_{3i+1}|_3))$$

such that $|Z_i| \geq \frac{|\alpha_{3i+1}|_3 - |\alpha_{3i}|_3}{z}$ and the function $t \mapsto f(t, Q_{|\alpha_p|_3}(|\alpha_{p-1}|_3), |\alpha_p|_3)$ is constant on Z_i, with a constant value, say $c_i < z$. This is possible because $P_{|\alpha_p|_3|\alpha_{p-1}|_3}$ is strictly increasing. On the other hand, $[p/3] - 1 > z$ implies $c_{i_0} = c_{i_1}$ for some i_0, i_1, with $i_0 < i_1 < [p/3] - 1$. We claim $Z_{i_0} \cup Z_{i_1}$ is a large homogeneous set for f. Note first that

$$|Z_{i_0} \cup Z_{i_1}| > |Z_{i_1}| \geq \frac{|\alpha_{3i_1+1}|_3 - |\alpha_{3i_1}|_3}{z} > |\alpha_{3i_1-1}|_3 > x > k\,.$$

Given $u, v, w \in Z_{i_0} \cup Z_{i_1}$ such that $u < v < w$, we have

$$f(u, v, w) = f(u, v, |\alpha_p|_3) = f(u, Q_{|\alpha_p|_3}(|\alpha_{p-1}|_3), |\alpha_p|_3)$$

because $\{u, v, w\} \subseteq \mathrm{Im}(Q_{|\alpha_p|_3})$. On the other hand,

$$f(u, Q_{|\alpha_p|_3}(|\alpha_{p-1}|_3), |\alpha_p|_3) = f(u', Q_{|\alpha_p|_3}(|\alpha_{p-1}|_3), |\alpha_p|_3)$$

for all $u' \in Z_{i_0} \cup Z_{i_1}$. Therefore, $Z_{i_0} \cup Z_{i_1}$ is f-homogeneous. The relative largeness follows because

$$\min(Z_{i_0} \cup Z_{i_1}) \leq \min(Z_{i_0}) < |\alpha_{3i_0+2}|_3 < \frac{|\alpha_{3i_1+1}|_3 - |\alpha_{3i_1}|_3}{z} \leq |Z_{i_1}| < |Z_{i_0} \cup Z_{i_1}|.$$

This completes the proof. □