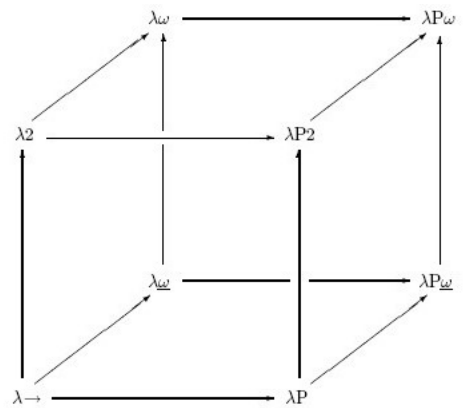
Summary of the Master thesis

Typed λ-claculus stems from the trial to avoid paradoxes which occurs in λ-calculus. The system $\lambda\!\rightarrow$-Curry of H. B. Curry is the first and simplest typed λ-calculus system. And Barendregt's λ-cube is the completion of the eight well-known typed λ-calculus systems which are in turn special examples of Pure Type Systems.



Barendregt's λ-cube

The systems deal with abstract algorithms. While all possible algorithms are allowed in λ-calculus, one can talk in typed λ-calculus only about certain algorithms, since the term construction is restricted by types. Types represent universes where mathematical objects live. A function from a type to another one can be applied to objects from the given type. The self-application of a algorithm is not allowed.

Another character of typed λ-calculus is that one can talk about logic. This is the idea of Curry-Howard-Isomorphism. And this can be realized by the relationship between the λ-cube and the logic-cube. It is very nice to see that the typed λ-calculus harmonizes with logic. The Calculus of Constructions (CC = le Calcul des Constructions = $\lambda P\omega$) of Th. Coquand and G. Huet is the strongest system in typed λ-calculus with which one can do the everyday mathematics. The investigation of the fine structure of this system is a possible approach to the λ-cube. The idea of Curry-Howard-Isomorphism plays the central role in it.

The starting point of this thesis is the question how strong the systems from the λ-cube are with respect to the representability of arithmetic functions. The functions which are representable in the λ-cube can be devided into three groups which are pairwise disjoint. The group of all functions representabl in $\lambda\!\rightarrow$ is so small that it contains just polynomially bounded functions. Even the predecessor function cannot be represented in the system $\lambda\!\rightarrow$. On the other hand, the other two groups have direct relationship with HA^2 (the second-order Heyting arithmetic) and HAH (the higher-order Heyting arithmetic). The functions representable in CC are the functions which are provably total in HA^2 and HAH respectively. There are arithmetical functions which are representable in CC, but not in $\lambda 2$.

This thesis consists of four chapters and is about representability of arithmetical function in each system from the λ-cube.

1. Introduction.

   The systems $\lambda\!\rightarrow$-Curry and $\lambda\!\rightarrow$-Church are introduced. And their basic properties are just mentioned since there are general arguments in the following chpters. The definition of representability is given, and it is shown which arithmetical functions are representable in $\lambda\!\rightarrow$-Church. All of these functions are polynomially bounded. One shows also that even the predecessor function is not representable.

2. PTSs and λ-cube.

   The concept PTS is a generalization of the λ-cube, while the λ-cube itself is a completion of certain typed λ-calculus systems. It comsists of eight subsystems of the Calculus of Constructions of Th. Coquand and G. Huet and contains the equivalent versions of $\lambda\!\rightarrow$-Church and Girard's F and $F\omega$. The 3-dimensional illustration of eight systems shows not only the subsystem relationship, but also the differences between every two systems. In the chapters 3 and 4 it is shown what kind of differences there are among the systems.

3. The Curry-Howard-Isomorphism and CC.

   The so-called logic-cube is introduced. Its relationship with the λ-cube is the idea of the Curry-Howard-Isomorphism. On the other hand, it is a possible approach to a better understanding of CC. The strong normalizability of $\lambda P2$ is proved. A direct generalization to CC can be carried out. The proof for $\lambda P2$ is so carried out that it can be formalized in the higher-order Heyting arithmetic HAH.

4. Representability in the λ-cube.

   The functions which are representable in the λ-cube are characterized. There are three groups which are pairwise disjoint. First, it is shown that the Ackermannian function is in $\lambda 2$ representable. One shows furthermore that the proof of strong normailizability of $\lambda 2$ in chapter 3 can be formalized in HAH, but not in HA^2. In order to show the non-conservativity of CC over $\lambda P2$, one carries out a particularized proof of the second Gödel's incompleteness theorem. This will be done with the help of the Curry-Howard-Isomorphism. Simultaneously it is shown that HAH is not Π^0_2 conservative over the second-order Heyting arithmetic HA^2. Finally, two arithmetical functions are given which are representable in $\lambda\omega$, but not in $\lambda 2$.